Laplace Distribution And Probabilistic (b_i) In Linear Programming Model

Ali Khaleel T. AL-Zubiadi
Ministry Of Higher Education And Scientific Research
Supervision & Scientific Evaluation

Abstract

The theory of probabilistic programming may be conceived in several different ways. As a method of programming it analyses the implications of probabilistic variations in the parameter space of linear or nonlinear programming model. The generating mechanism of such probabilistic variations in the economic models may be due to incomplete information about changes in demand, production and technology, specification errors about the econometric relations presumed for different economic agents, uncertainty of various sorts and the consequences of imperfect aggregation or disaggregating of economic variables. In this Research we discuss the probabilistic programming problem when the coefficient b_i is random variable with given Laplace distribution.

[4]*1-1: Linear Programming*

Linear Programming often represents allocation problem in which limited resources are allocated to a number of economic activities.
To provide this interpretation we write the LP model as follows:

$$Max(or\ Mim)\quad Z = c_1\chi_1 + c_2\chi_2 + \dots\dots\dots + c_n\chi_n$$

$$subject\ to$$

$$\begin{aligned}
&a_{11}\chi_1 + a_{12}\chi_2 + \dots\dots\dots + a_{1n}\chi_n \le \quad (or \ge) \quad b_1 \\
&a_{21}\chi_1 + a_{22}\chi_2 + \dots\dots\dots + a_{2n}\chi_n \le \quad (or \ge) \quad b_2 \\
&. \\
&. \\
&. \\
&. \\
&a_{m1}\chi_1 + a_{m2}\chi_2 + \dots\dots\dots + a_{mn}\chi_n \le \quad (or \ge) b_m \\
&\quad \chi_1, \chi_2, \chi_3 \dots\dots\dots \chi_n \ge 0
\end{aligned}$$

From the economic standpoint ,linear programming seeks the best allocation of limited resources to specific economic activities.
In the general LP model , there are n activities whose levels are
$\chi_1, \chi_2, ... \chi_n$ **represented by**
There are also m resources whose maximum or minimum availabilities are given by
$b_1, b_2, b_3, ..., b_m$

each unit of activity j consumes an amount a_{ij} of resource i .this means that the quantity $\sum_{j=1}^{n} a_{ij} \chi_j$ **represents the total usage by all n activities of resource i and hence cannot exceed b_i .**
The objective function $\sum_{j=1}^{n} c_j \chi_j$ **represents a measure of the contribution of the different activities . In the maximization case , c_j represents the profit per unit of activity j , whereas in the case of minimization , c_j represents the cost per unit .**
We note that the worth of an activity cannot be judged in terms of the objective coefficient c_j only , the activity's consumption of the limited resources is also an important factor , because all the activities of the model are competing for limited resources , the relative contribution of an activity depends on the both its objective coefficient c_j and its consumption of the resources a_{ij} .
Thus an activity with very high unit profit may remain at the zero level because of its excessive use of limited resources.

1-2: Probabilistic Linear Program[1]

A stochastic or probabilistic program is a programming problem in which some or all of the problem data is random. For such a program, we must therefore define the concepts of feasible and optimal solutions that will properly account for the random nature of the problem. A stochastic linear program can be written as:

$$Min \quad Z = \sum_{j=1}^{n} c_j \chi_j$$

$$S.t$$

$$\sum_{j=1}^{n} a_{ij} \chi_j \geq b_i \qquad (i = 1, ..., m)$$

$$\chi_j \geq 0 \qquad (j = 1, ..., n)$$

where the coefficients c_j, a_{ij} and b_i are random variables with given probability distributions.

There are two types of decision rules for determining the optimal values of the decision variables χ_i. The type of decision rules that determine the optimal values of χ_j before the actual" values of the random elements become known are called zero order rules. The other type of decision rules are known as nonzero order rules. In these rules, we wait for the values of the random elements to become known before determining χ_i, but decide in advance how the knowledge of the sample values of the random elements is going to be used.

The unknown values of the decision variables may be assumed deterministic. If this is the case, a decision rule is called a nonrandomized decision rule. Since the random variations in the parameters of a problem induce random variations in the optimal values of variables χ_i, we can have a chance mechanism to determine the optimal values of χ_i. The rules governing such a mechanism are called randomized decision rules. In this case, χ_j are treated as random variables, and consequently we may find their probability distributions.

Now, we turn our attention to an important class of stochastic programming problems, called the chance-constrained problems. These problems were initially studied by A. Charnes and W. W. Cooper. In a stochastic programming problem, some constraints may be deterministic and the remaining may involve random elements. On the other hand, in a chance-constrained programming problem, the latter set of constraints is not required to always hold, but these must hold simultaneously or individually with given probabilities. In other words, we are given asset of probability measures indicating the extent of violation of the random constraints, The general chance-constrained linear program is of the form:

$$Min \quad Z = \sum_{j=1}^{n} c_j \, \chi_j$$

$$S.t$$

$$\sum_{j=1}^{n} a_{ij} \, \chi_j \geq b_i \qquad (i = 1, \ldots\ldots\ldots, m_1) \qquad \ldots\ldots\ldots\ldots(1)$$

$$P_r \left(\sum a_{ij} \chi_j \geq b_i \right) \geq u_i \quad (i = m_1 + 1, \ldots\ldots\ldots, m) \qquad \ldots\ldots\ldots\ldots(2)$$

$$\chi_j \geq 0 \qquad (j = 1, \ldots\ldots\ldots\ldots, n)$$

where, $0 < u_i < 1$. for all i= $m_1+1, \ldots\ldots m$, and Pr( ) means the probability of the event in parentheses.

Constraints (1) are deterministic in the sense that the coefficients involved are deterministic.

Constraints (2) are Stochastic can be violated with a probability less than $1-u_i$

1-3:Deterministic Equivalents of Probabilistic[1] Constraints

In program for above, constraints (1) are deterministic, and thus need not be altered. Further, the chance constraints given by (2), i.e.,

$$P_r\left(\sum a_{ij}\chi_j \geq b_i\right) \geq u_i \quad (i = m_1 + 1, \ldots\ldots, m) \qquad \ldots\ldots\ldots\ldots(2)$$

are not deterministic. We shall therefore find their deterministic equivalents: Only. b_i are random variables. Let the distribution function of b_i be:

$$F_{bi}(z) = P_r(b_i \leq z)$$

Then:

$$F_{bi}(z) \geq u_i$$

$$\sum_{j=1}^{n} a_{ij}\chi_j \geq F_{bi}^{-1}(u_i) = \tau_u \qquad \ldots\ldots\ldots\ldots(3)$$

Equation (3) is equivalent to the nonstochastic linear constraint.

1-4: Laplace Distribution[2,3]

This is a continuous probability distribution. It is named after a French mathematician. Wikipedia points out that it is also known as a double exponential distribution, because it reminds one of an exponential distribution "spliced together back-to-back."

This distribution is characterized by location parameter φ (any real number) and scale parameter λ (has to be greater than zero) parameters.

The probability density function of Laplace(φ,λ) is:

$$f(\chi|\varphi, \lambda) = \frac{1}{2\lambda}\exp\left(-\frac{|\chi - \varphi|}{\lambda}\right)$$

The cumulative density function looks even more impressive, yet rather easy to integrate because of the absolute value in the formula:

$$F(\chi|\varphi, \lambda) = \frac{1}{2}\exp\left(\frac{\chi - \varphi}{\lambda}\right), \quad when(\chi \leq \varphi)$$

and

$$F(\chi|\varphi, \lambda) = 1 - \frac{1}{2}\exp\left(\frac{\varphi - \chi}{\lambda}\right), \quad when(\chi \succ \varphi)$$

The exponential distribution's probability density function is defined for $\chi > 0$:

$$Exponential(\frac{1}{\lambda}):\ f(\chi|\lambda) = \frac{1}{\lambda}\exp(-\frac{\chi}{\lambda}),\quad \chi \succ 0$$

Unlike the exponential, the Laplace is defined $-\infty < \chi < \infty$.

If $\varphi = 0$, then the probability density function for Laplace on $\chi > 0$ is equal to 1/2 of the probability of the exponential. In Figure (1,2) , we illustrate this fact by plotting the probability density of the Laplace on (—15,15) side-by-side with an exponential distribution, and one can observe that if the exponential is divided by half, then it is equal to the Laplace:

Figure 1 : Laplace p.d.f where φ =0 , λ=1

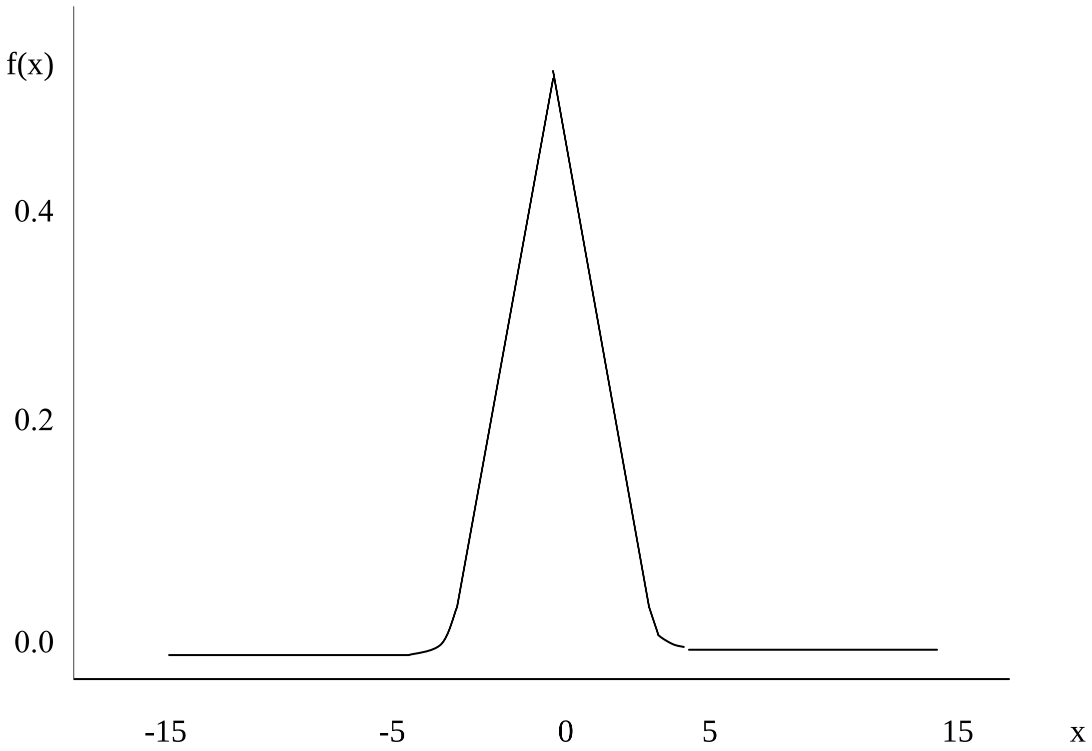
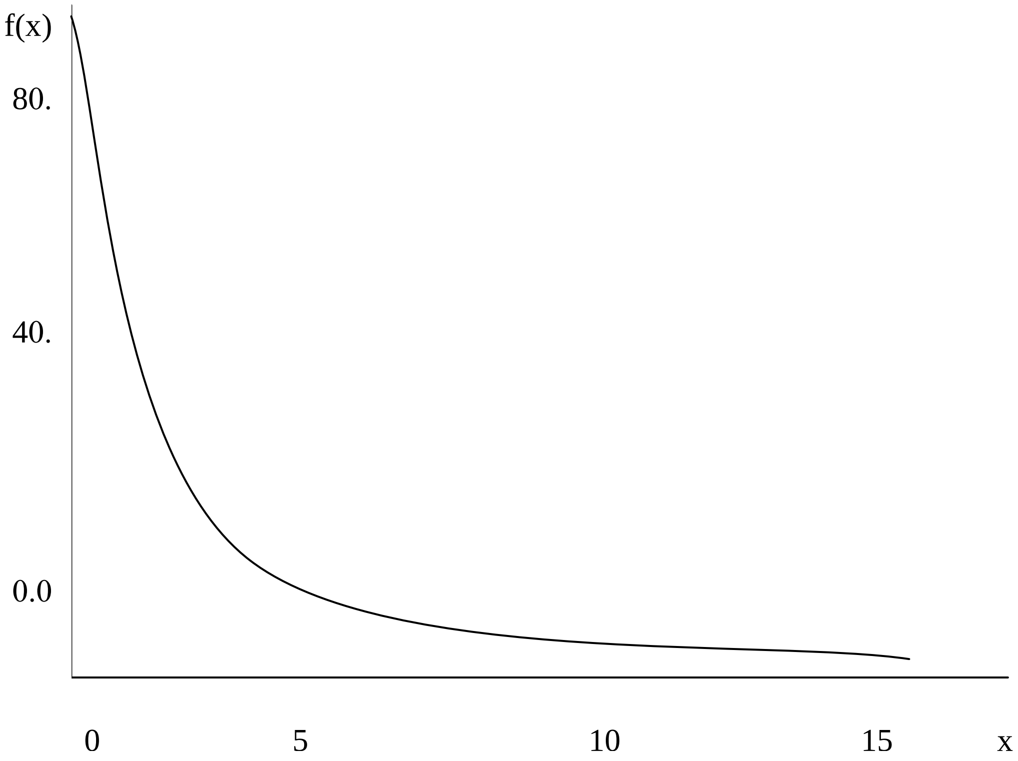


Figure 2:Exponential p.d.f where λ=1



The expected value of a Laplace distribution is:

$$E(x) = \varphi$$

As in the case of other symmetrical distributions, such as the Normal and the logistic distributions. Laplace's location is the same as its mean, median, and mode. The variance is:

$$Var(\chi) = 2\lambda^2$$

From the Figure 3, we see that the scale parameter determines the width of the distribution. From Figure 4, it is apparent that changing the location simply shifts the probability density curve to the right or to the left.

Figure3: Laplace Density Function where φ =4

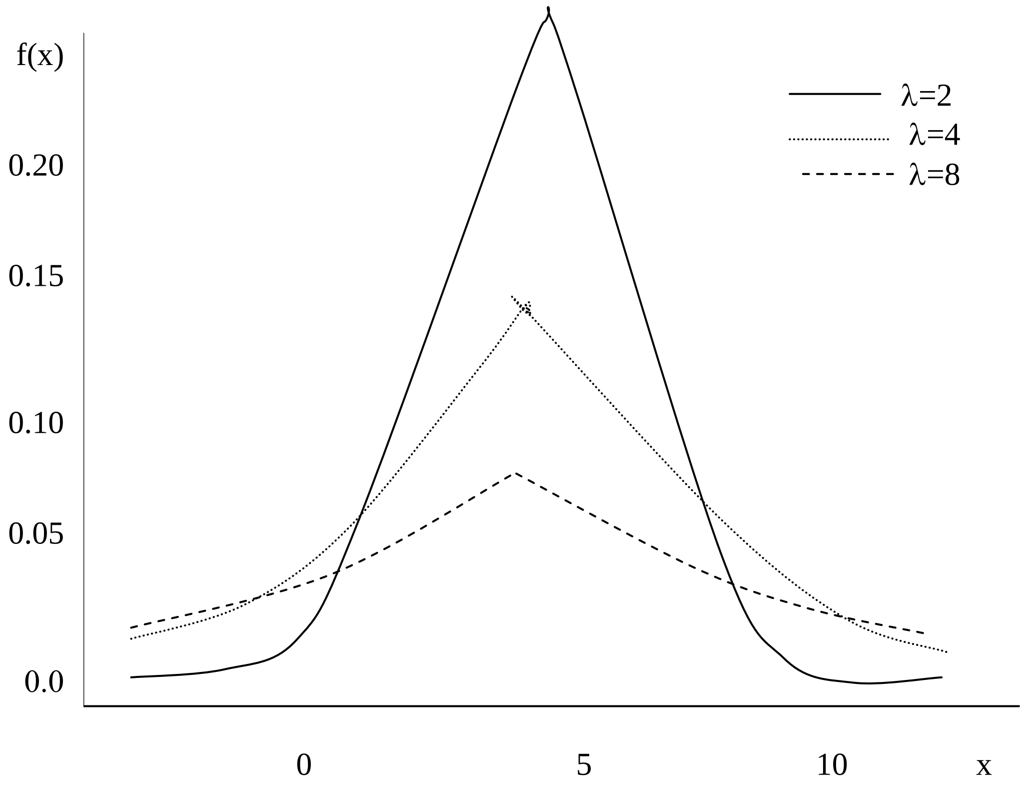
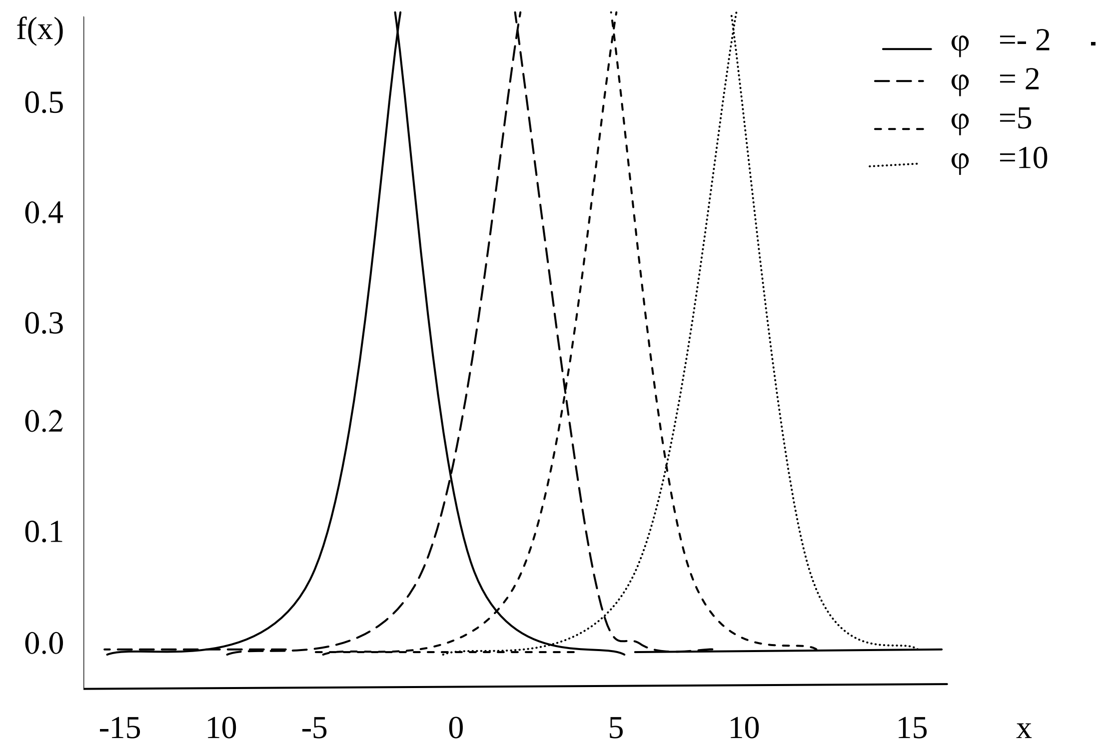


Figure 4: Laplace Density Function where λ =1



1-5: Illustratory Example

$$Max\,Z = 5\chi_1 + 4\chi_2$$

$S.T$

$$P_r(3\chi_1 + 5\chi_2 \geq b_1) \geq 0.60$$
$$P_r(0.2\chi_1 + 0.3\chi_2 \leq b_2) \geq 0.40$$
$$P_r(\chi_1 + \chi_2 \geq b_3) \geq 0.30$$
$$\chi_1, \chi_2, \chi_3 \geq 0$$

$$b_1 \approx Lap(2,3)$$
$$b_2 \approx Lap(1,4)$$
$$b_3 \approx Exp(3)$$

***The Probabilistic Constraint** $P_r(3\chi_1 + 5\chi_2 \geq b_1) \geq 0.60$ **is equivalent:**

$$3\chi_1 + 5\chi_2 \geq \tau_u$$

Where :

$$\int_{-\infty}^{\tau_u} \frac{1}{2\lambda} \exp\left(\frac{\chi - \varphi}{\lambda}\right) d\chi = 0.60$$

$$\int_{-\infty}^{\tau_u} \frac{1}{6} \exp\left(\frac{\chi - 2}{3}\right) d\chi = 0.60$$

$$\frac{1}{2}\left[\ \exp\left(\frac{\chi - 2}{3}\right)\ \right]_{-\infty}^{\tau_u} = 0.60$$

$$\frac{1}{2}\left[\ \exp\left(\frac{\tau_u - 2}{3}\right) - 0\right] = 0.60$$

$$Ln\left(\frac{1}{2}\right) + \frac{\tau_u - 2}{3} = Ln0.60$$

$$\tau_u = 3(Ln0.60 - Ln0.5) + 2$$

$$\tau_u = 4.453$$

Then the deterministic Constraint is: $3\chi_1 + 5\chi_2 \geq 4.453$

***The Probabilistic Constraint** $P_r(0.2\chi_1 + 0.3\chi_2 \le b_2) \ge 0.40$ **is equivalent :**

$$0.2\chi_1 + 0.3\chi_2 \le F^{-1}(1-u) = \tau_N$$

Where :

$$\int_{\tau_N}^{\infty_u} \frac{1}{2\lambda}\exp(-\frac{\chi-\varphi}{\lambda})d\chi = 0.40$$

$$\int_{\tau_N}^{\infty} \frac{1}{8}\exp(-\frac{\chi-1}{4})d\chi = 0.40$$

$$-\frac{1}{2}[\ \exp(-\frac{\chi-1}{4})\ ]_{\tau_N}^{\infty} = 0.40$$

$$-\frac{1}{2}\left[0-\exp(-\frac{\tau_N-1}{4})\ \right] = 0.40$$

$$Ln\left(\frac{1}{2}\right) - \frac{\tau_N - 1}{4} = Ln0.40$$

$$\tau_N = 4(Ln(1/2) - Ln0.40) + 1$$

$$\tau_N = 3.65$$

Then the deterministic Constraint is: $0.2\chi_1 + 0.3\chi_2 \le 3.65$

***The Probabilistic Constraint** $P_r(\chi_1 + \chi_2 \ge b_3) \ge 0.30$ **is equivalent :**

$$\chi_1 + \chi_2 \ge \tau_u$$

$$b_3 \approx Exp(3) = 2Lap(0,3)$$

Where :

$$\int_{-\infty}^{\tau_u} \frac{1}{\lambda}\exp(\frac{\chi}{\lambda})d\chi = 0.30$$

$$\int_{-\infty}^{\tau_u} \frac{1}{3}\exp(\frac{\chi}{3})d\chi = 0.30$$

$$[\ \exp(\frac{\chi}{3})\ ]_{-\infty}^{\tau_u} = 0.30$$

$$\left[\ \exp(\frac{\tau_u}{3}) - 0\right] = 0.30$$

$$\frac{\tau_u}{3} = Ln0.30$$

$$\tau_u = 3(Ln0.30)$$

$$\tau_u = 5.985$$

Then the deterministic Constraint is: $\chi_1 + \chi_2 \geq 5.985$

The deterministic Programming Problem is:

$$
\begin{aligned}
&Max\, Z = 5\chi_1 + 4\chi_2 \\
&S.T \\
&3\chi_1 + 5\chi_2 \geq 4.453 \\
&0.2\chi_1 + 0.3\chi_2 \leq 3.65 \\
&\chi_1 + \chi_2 \geq 5.985 \\
&\qquad \chi_{1,}\chi_2, \chi_3 \geq 0
\end{aligned}
$$

Conclusions

1. **Linear deterministic Programming often represents allocation problem in which limited resources are allocated to a number of economic activities.**
2. **Probabilistic program is a programming problem in which some or all of the problem data is random , may be due to incomplete information about changes in demand, production and technology.**
3. **Probabilistic programming is the most realistic of deterministic programming dealing with problems of life.**
4. **solving Probabilistic programming problem need to know mathematical distribution to random parameter.**
5. **Laplace distribution is known as a double exponential distribution, because it reminds one of an exponential distribution "spliced together back-to-back".**

References

1. **Kambo,N.s. -1979-Mathematical Programming, Affiliated East-West Press Private Limited.**
2. **Nadarajah , R -2004- " Reliability for Laplace Distribution", math. Prob. Eng.10 ,pp.169-183**
3. **Stockute, raminta & Johnson, paul -2006- Laplace Distribution (http://pj.freefaculty.org).**
4. **Taha, Hamady A. -1999- Operation Research, Macmillan Publishing Company (New york).**