Bayes Estimators for the Parameter of the Inverted Exponential Distribution Under different Double informative priors

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Abstract

In this paper, we present a comparison of double informative priors which are assumed for the parameter of inverted exponential distribution.To estimate the parameter of inverted exponential distribution by using Bayes estimation ,will be used two different kind of information in the Bayes estimation; two different priors have been selected for the parameter of inverted exponential distribution. Also assumed Chi-squared - Gamma distribution, Chi-squared - Erlang distribution, and- Gamma- Erlang distribution as double priors. The results are the derivations of these estimators under the squared error loss function with three different double priors.

Additionally Maximum likelihood estimation method (MLE) was used to estimate the parameter of inverted exponential distribution .We used simulation technique, to compare the performance for each estimator, several cases from inverted exponential distribution for data generating, for different samples sizes (small, medium, and large).Simulation results shown that the best method is the bayes estimation according to the smallest values of mean square errors(MSE) for all samples sizes (n) comparative to the estimated values by using MLE . According to obtained results, we see that when the double prior distribution for θ is Gamma- Erlang distribution for some values for the parameters a, b & λ given the best results according to the smallest values of mean square errors (MSE) comparative to the same values which obtained by using Maximum likelihood estimation (MLE) for the assuming true values for $\theta = 1, 1.5, 2.5$ and for all samples sizes.

Key words/ Inverted exponential distribution, Bayes method, Prior distributions (Chi-squared distribution, Gamma distribution, Erlang distribution), mean squared errors (MSE).

Introduction

The Inverted Exponential Distribution (IED) is a member of continuous probability distributions. It has been introduced by Keller and Kamath [4] in (1982). The inverted exponential distribution is studied as a prospective life distribution. We mention some of studies in a brief manner: In (2007) Dey [3] derived Bayes' estimators for nine of the parameter of inverted exponential distribution. His results are the derivations of these estimators based on the squared error loss function and LINEX loss functions.

In (2009) Prakash[6] discussed the properties of the bayes estimator, shrinkage estimator and minimax estimator of the parameter under the squared error loss function and GELF for the inverted exponential distribution. He also presented the moments of the lower record value and the estimation of the parameter, based on a series of observed record values by the maximum likelihood and moment methods.In (2012) he [7] investigated the properties of Bayes estimators of the parameter, reliability function and hazard rate under the symmetric and asymmetric loss functions for the inverted exponential model. And he determined the Bayes predictive interval and the Bayes estimate of shift point.

In (2015) Nadia & Suzan[5] used Bayes estimation method to estimate the scale parameter of the inverted exponential distribution and the maximum likelihood method. They obtained bayes estimators based on symmetric "squared error" and asymmetric "precautionary" loss functions corresponding to informative "inverted gamma and Gumbel type II" and non-informative "Jeffrey and extension of Jeffrey" priors.

The Aim of this paper is to obtain bayes Estimators for the parameter of the inverted exponential distribution under different double informative priors. A few studies present in double informative priors, we mention it: Abdul Haq, Muhammad Aslam[1], they used double prior selection for discrete case in the case of Poisson distribution. Radha and Vekatesan (2015), they studied double prior selection for continuous case in the case of Maxwell distribution[8] . They have assumed generalized uniform-inverted Gamma distribution as double priors.

Here we study double prior selection for continuous case in the case of the inverted exponential distribution. We have assumed Chi-squared - Gamma distribution, Chi-squared - Erlang distribution, and- Gamma- Erlang distribution as double priors. Also we used the maximum likelihood estimator. We try to find best method to estimate parameter of inverted exponential distribution .According to the smallest value of Mean Square Errors (MSE) were calculated to compare the methods of estimation. Several cases from inverted exponential distribution for data generating, of different samples sizes (small, medium, and large).The results were obtained by using simulation technique.

1. The Inverted Exponential Distribution

Let us consider $t_1, t_2, \ldots, t_n$ is a random sample of n independent observations from an inverted exponential distribution (IED) having the probability density function (pdf) define as [9]:

$$f(t;\theta) = \frac{\theta}{t^2}\exp(-\frac{\theta}{t}) \quad , \quad t \geq 0 \ , \theta > 0 \qquad \ldots (1)$$

And the cumulative distribution function (cdf) is given as;

$$F(t;\theta) = \exp(-\frac{\theta}{t}) \quad , \quad t \geq 0 \ , \theta > 0 \qquad \ldots (2)$$

The r th moment about origin is defined as:

$$M_r = E(t^r) = \int_0^\infty t^r \ \frac{\theta}{t^2}\exp(-\frac{\theta}{t})\,dt = \theta^r \Gamma(1-r) \qquad \ldots (3)$$

This integral exists only for (r < 1).

2. Parameter Estimation Methods

In this section, we used Maximum likelihood Estimation and bayes Estimation to estimation parameter θ under different double informative priors.

3.1 Maximum Likelihood Estimation (MLE)

From the inverted exponential pdf given in (1) the likelihood function will be as follows[2]:

$$L(\underline{t} \backslash \theta) = \prod_{i=1}^{n} f(t;\theta) = \frac{1}{\prod_{i=1}^{n} t_i^2}\theta^n \exp(-\theta\sum_{i=1}^{n}\frac{1}{t_i}) \qquad \ldots (4)$$

By taking the log and differentiating partially with respect to θ, we get:

$$\frac{\partial}{\partial\theta}\log L(\underline{t} \backslash \theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{1}{t_i} \qquad \ldots (5)$$

Then the MLE of θ is the solution of equation (5) after equating the first derivative to zero, Hence:

$$\hat{\theta}_{MLE} = \frac{n_i}{\sum_{i=1}^{n}\frac{1}{t_i}} \qquad \ldots (6)$$

3.2 Bayes Estimation Method

Let $t_1, t_2, ..., t_n$ be a random sample of size n with probability density function given in equation (1) and likelihood function given in equation (4).In this paper the posterior distributions for the unknown parameter θ are derived under different double informative priors. Here we have assumed Chi-squared distribution [10] – Gamma distribution[12], Chi-squared distribution – Erlang distribution [11] distribution and Gamma- Erlang distribution as double priors, to get bayes estimation.

3.3.1 The posterior distribution using different double priors

It is assumed that θ follows three types of prior distributions with pdf as given in table -1:

Table -1: The three types of prior distributions ($f_i(\theta)$) with pdf for θ.

Prior distribution	$f_i(\theta) \quad , i = 1, 2, 3$
θ **~ Chi-square(** v **)**	$f_1(\theta) = \frac{1}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{-\frac{v}{2}} \theta^{\frac{v}{2}-1} \exp(-\frac{1}{2}\theta)$ for $\theta \geq 0$, $v = 1, 2, \; ....v$
θ **~Gamma(a ,** b **)**	$f_2(\theta) = \frac{b^a}{\Gamma a} \theta^{a-1} \exp(-b\theta)$ for $\theta \geq 0$, $a, b > 0$
θ **~erlang (** λ **)**	$f_3(\theta) = \lambda^2 \; \theta \; \exp(-\lambda\theta)$ for $\theta \geq 0$, $\lambda > 0$

And their double prior's distributions with pdf as given in table -2:

Table -2: The three types of double prior distributions ($f_i(\theta)$) with pdf for θ.

Prior distribution	$P_i(\theta) \quad , i = 1, 2, 3$
θ **~ Chi-square(** v **)-** θ **~Gamma(a ,** b **)**	$P_1(\theta) \; \alpha \; f_1(\theta) \; f_2(\theta)$ $P_1(\theta) \; \alpha [\frac{1}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{-\frac{v}{2}} \frac{b^a}{\Gamma a}] \; \theta^{\frac{v}{2}+a-2} \exp(-\theta(\frac{1}{2} + b))$
θ **~ Chi-square(** v **)-** θ **~Erlang (** λ **)**	$P_2(\theta) \; \alpha \; f_1(\theta) \; f_3(\theta)$ $P_2(\theta) \; \alpha [\frac{\lambda^2}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{-\frac{v}{2}}] \; \theta^{\frac{v}{2}+1-1} \exp(-\theta(\frac{1}{2} + \lambda))$
θ **~Gamma(a ,** b **)-** θ **~Erlang (** λ **)**	$P_3(\theta) \; \alpha \; f_2(\theta) \; f_3(\theta)$ $P_3(\theta) \; \alpha [\frac{b^a}{\Gamma a} \; \lambda^2] \; \theta^a \exp(-\theta(b + \lambda))$

Then the posterior distribution of θ for the given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is given by:

$$P(\theta \backslash t)=\frac{L(\underline{t}\backslash\theta)\;P(\theta)}{\int_\theta L(\underline{t}\backslash\theta)\;P(\theta)d\theta} \quad \ldots(7)$$

Substituting the equation (4) and for each $P(\theta)$ as shown in table -2 in equation (7), we get the posterior distributions for the unknown parameter θ are derived using the following three types of double priors (for more details see Appendix-A).

Table -3: The posterior distributions ($P(\theta\backslash t)$) for the unknown parameter (θ) are derived using the following three types of priors.

Double prior distⁿ.	The posterior distribution ($P(\theta\backslash t)$)
$\theta\sim$ Chi-square(v)- $\theta\sim$Gamma(a , b)	$P_1(\theta\backslash t))\sim$ Gamma distribution ($a_{(new)}=(n+a+\frac{v}{2}-1),\ b_{(new)}=(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)$) $P_1(\theta\backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)}\theta^{(n+\frac{v}{2}+a-1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b))$
$\theta\sim$ Chi-square(v)- $\theta\sim$Erlang (λ)	$P_2(\theta\backslash t))\sim$ Gamma distribution ($a_{(new)}=(n+\frac{v}{2}+1),\ b_{(new)}=(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)$) $P_2(\theta\backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\theta^{(n+\frac{v}{2}+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)),\ n,v,\lambda>0$
$\theta\sim$Gamma(a , b)- $\theta\sim$Erlang (λ)	$P_3(\theta\backslash t))\sim$ Gamma distribution ($a_{(new)}=(n+a+1),\ b_{(new)}=(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)$) $P_3(\theta\backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)^{(n+a+1)}}{\Gamma(n+a+1)}\theta^{(n+a+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)),\ a,b,n,\lambda>0$

3.3.2 <u>Bayes' Estimators</u>

Bayes' estimators for the parameter θ, was considered with three different double priors and under the squared error loss function $L_1(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2$.Where $\hat{\theta}$ is an estimator for θ, was considered with three different double priors, and under the squared error loss function . Following is the derivation of these estimators:

3.3.2.1 The squared error loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$$L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \qquad \dots (8)$$

After simplified steps, we get Bayes estimator of θ denoted by $\hat{\theta}_{SE}$ for the above prior as follows

$$\hat{\theta}_{SE} = E(\theta \setminus t) = \int_0^\infty \theta \, P(\theta \setminus t) \, d\theta \qquad \dots (9)$$

So, the following results are the derivations of these estimators under the squared error loss function with three different double priors (for more details see Appendix-B).

Table -4: The estimators ($\hat{\theta}_{SE}$) under the squared error loss function With three different double priors.

Double Prior distribution	$\hat{\theta}_{SE} = E(\theta \setminus t) = \int_0^\infty \theta \, P(\theta \setminus t) \, d\theta$
θ **~ Chi-square(** v **)-** θ **~Gamma(a ,** b **)**	$\hat{\theta}_{SE1} = \dfrac{\Gamma(n+a+\frac{v}{2})}{\Gamma(n+a+\frac{v}{2}-1)\ (\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + b)}$, $n, b, a, v > 0$
θ **~ Chi-square(** v **)-** θ **~Erlang (** λ **)**	$\hat{\theta}_{SE2} = \dfrac{\Gamma(n+\frac{v}{2}+2)}{\Gamma(n+\frac{v}{2}+1)\ (\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + \lambda)}$, $n, v, \lambda > 0$
θ **~Gamma(a ,** b **)-** θ **~Erlang (** λ **)**	$\hat{\theta}_{SE3} = \dfrac{\Gamma(n+a+2)}{\Gamma(n+a+1)\ (\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda)}$, $n, b, a, \lambda > 0$

4. Simulation Study

In this study, we have generated random samples from inverted exponential distribution and compared the performance of MLE and bayes estimator based on them. So we have considered several steps to perform simulation study as follow:

1. **We have chosen sample size n = 30, 60, 90 and 120 to represent small, moderate and large sample size.**

2.**We generated data from inverted exponential distribution for the scale parameter, according to the following cdf** $F(t;\theta)=\exp(-\frac{\theta}{t})$ **, by setting** $F_i = u_i$ **where** $u_i \sim \text{uniform dist.}^n(0,1)$**, we have** $t_i = -(\theta/\ln(u_i))$**.We have considered randomly several values for the parameter of exponential distribution** $\theta = 0.5,\ 1,1.5,\ 2.5$**.**

3.**We used randomly the values for the parameters of the Chi-square(**v**)-Gamma(a ,**b**) distribution (v= 1, 2) and (a , b) = (0.5,1) ,(1,2),(2,2),(2,1) as double prior distribution for**θ**.**

4.**We used randomly the values for the parameters of the Chi-square (**v**)-Erlang (**λ**) distribution (v=1, 2) and (**λ **=0.5, 1, 2) as double prior distribution for**θ**.**

5.**We used randomly the values for the parameters of the Gamma(a ,**b**)-Erlang (**λ**) distribution (a , b) = (0.5,1) ,(1,2),(2,2),(2,1) and (**λ **=0.5,1,2) as double prior distribution for**θ**.**

6.**The number of replication used was (**$r = 1000$**) for each sample size (n).**

We obtained estimators for the parameter from equations (6) and the estimators in table -4, it means the estimators $(\hat{\theta}_{SE})$ **under the squared error loss function with three different double priors .The simulation program was written by using MATLAB-R2008a program. After the parameter** θ **was estimated, Mean Square Errors (MSE) was calculated to compare the methods of estimation, where:**

$$MSE = \frac{1}{r}\sum_{r=1}^{1000}(\hat{\theta}(r) - \theta)^2 \qquad ...(10)$$

See appendix-C for the programs algorithm. The results of the simulation study are summarized and tabulated in tables (4-1) - (4-4).In each row of tables (4-1) – (4-4) ,we have four estimated values for θ $(\hat{\theta})$ **with MSE for all samples sizes (n) and values (v , a , b,** λ**) respectively. By using different estimation methods that is maximum likelihood estimator and the Bayes estimators in three types of double priors distributions. So our criterion is the best method that gives the smallest value of (MSE). We list the results in the following tables (4-1) – (4-4).**

4-1: MSE of estimated inverted exponential parameter under squared error loss function.

method	parameters				Estimate for θ ($\hat{\theta}$)				MSE			
					Sample Size(n)				Sample Size(n)			
	θ	-	-	-	30	60	90	120	30	60	90	120
MLE	0.5				0.51281	0.50889	0.50505	0.50543	0.00866	0.00444	0.00295	0.00238
Bayes	θ	v	a	b	$(P_1(\theta \backslash x))$ = Chi-square - Gamma distribution							
	0.5	1	0.5	1	0.49959	0.5024	0.50079	0.50223	0.00763	0.00415	0.00283	0.00229
			1	2	0.49936	0.50231	0.50075	0.5022	0.00736	0.00407	0.00279	0.00227
			2	2	0.51573	0.51061	0.50628	0.50637	0.0081	0.00432	0.00289	0.00235
			2	1	0.52457	0.51496	0.50913	0.50851	0.00902	0.00457	0.00300	0.00242
		2	0.5	1	0.50792	0.50658	0.50357	0.50432	0.00795	0.00425	0.00287	0.00233
			1	2	0.50754	0.50646	0.50352	0.50429	0.00766	0.00418	0.00284	0.00231
			2	2	0.52391	0.51476	0.50905	0.50845	0.00867	0.00449	0.00297	0.00240
			2	1	0.5329	0.51914	0.51192	0.5106	0.00977	0.00479	0.00309	0.00248
Bayes	θ	v	λ	-	$(P_2(\theta \backslash x))$ = Chi-square - Erlang distribution							
	0.5	1	0.5		0.52912	0.51716	0.51057	0.50958	0.00956	0.00472	0.00307	0.00246
			1		0.52457	0.51496	0.50913	0.50851	0.00902	0.00457	0.00300	0.00242
			2		0.51573	0.51061	0.50628	0.50637	0.0081	0.00432	0.00289	0.00235
		2	0.5		0.53751	0.52136	0.51336	0.51168	0.01040	0.00496	0.00316	0.00252
			1		0.5329	0.51914	0.51192	0.5106	0.00977	0.00479	0.00309	0.00248
			2		0.52391	0.51476	0.50905	0.50845	0.00867	0.00449	0.00297	0.00240
Bayes	θ	a	b	λ	$(P_3(\theta \backslash x))$ = Gamma - Erlang distribution							
	0.5	0.5	1	0.5	0.52457	0.51496	0.50913	0.50851	0.00902	0.00457	0.00300	0.00242
		1	2		0.52391	0.51476	0.50905	0.50845	0.00867	0.00449	0.00297	0.00240
		2	2		0.54029	0.52306	0.51458	0.51262	0.01024	0.00494	0.00316	0.00252
		2	1		0.54955	0.52752	0.51748	0.51479	0.01169	0.00532	0.00332	0.00262
		0.5	1	1	0.52011	0.51277	0.5077	0.50744	0.00853	0.00444	0.00295	0.00238
		1	2		0.51954	0.5126	0.50763	0.50739	0.00821	0.00436	0.00291	0.00236
		2	2		0.53577	0.52087	0.51315	0.51154	0.00960	0.00477	0.00309	0.00248
		2	1		0.54488	0.52528	0.51603	0.5137	0.01093	0.00513	0.00324	0.00257
		0.5	1	2	0.51142	0.50847	0.50487	0.50531	0.00771	0.00420	0.00285	0.00232
		1	2		0.511	0.50833	0.50482	0.50527	0.00743	0.00413	0.00281	0.00230
		2	2		0.52697	0.51653	0.5103	0.50941	0.00851	0.00447	0.00296	0.00239
		2	1		0.53577	0.52087	0.51315	0.51154	0.00960	0.00477	0.00309	0.00248

4-2: MSE of estimated inverted exponential parameter under squared error loss function.

method	parameters				Estimate for θ ($\hat{\theta}$)				MSE			
					Sample Size(n)				Sample Size(n)			
	θ	-	-	-	30	60	90	120	30	60	90	120
MLE	1				1.0256	1.0178	1.0145	1.0081	0.03464	0.01778	0.01328	0.00871
Bayes	θ	v	a	b	$(P_1(\theta \backslash x))$ = Chi-square - Gamma distribution							
	1	1	0.5	1	0.97413	0.99213	0.99744	0.99541	0.02819	0.01581	0.01221	0.00824
			1	2	0.95837	0.98387	0.99186	0.99127	0.02661	0.01523	0.01186	0.00809
			2	2	0.9898	1.0001	1.0028	0.99949	0.02664	0.01547	0.01206	0.00815
			2	1	1.0228	1.0169	1.0141	1.0079	0.03087	0.01683	0.01281	0.00849
		2	0.5	1	0.99037	1.0004	1.003	0.99956	0.02854	0.01601	0.01235	0.00829
			1	2	0.97409	0.992	0.99734	0.99538	0.02638	0.01528	0.01193	0.00810
			2	2	1.0055	1.0083	1.0083	1.0036	0.02742	0.01579	0.01226	0.00823
			2	1	1.0391	1.0252	1.0196	1.012	0.03284	0.01745	0.01313	0.00864
Bayes	θ	v	λ	-	$(P_2(\theta \backslash x))$ = Chi-square - Erlang distribution							
	1	1	0.5		1.0402	1.0255	1.0198	1.0121	0.03413	0.01778	0.01329	0.00872
			1		1.0228	1.0169	1.0141	1.0079	0.03087	0.01683	0.01281	0.00849
			2		0.9898	1.0001	1.0028	0.99949	0.02664	0.01547	0.01206	0.00815
		2	0.5		1.0567	1.0339	1.0254	1.0162	0.03677	0.01855	0.01369	0.00890
			1		1.0391	1.0252	1.0196	1.012	0.03284	0.01745	0.01313	0.00864
			2		1.0055	1.0083	1.0083	1.0036	0.02742	0.01579	0.01226	0.00823
Bayes	θ	a	b	λ	$(P_3(\theta \backslash x))$ = Gamma - Erlang distribution							
	1	0.5	1	0.5	1.0228	1.0169	1.0141	1.0079	0.03087	0.01683	0.01281	0.00849
		1	2		1.0055	1.0083	1.0083	1.0036	0.02742	0.01579	0.01226	0.00823
		2	2		1.0369	1.0245	1.0193	1.0118	0.03049	0.01684	0.01283	0.00849
		2	1		1.0715	1.0417	1.0307	1.0203	0.03842	0.01911	0.01397	0.00905
		0.5	1	1	1.006	1.0085	1.0084	1.0037	0.02840	0.01607	0.01240	0.00830
		1	2		0.98955	1.0000	1.0027	0.99946	0.02577	0.01521	0.01193	0.00808
		2	2		1.0205	1.0161	1.0136	1.0077	0.02771	0.01596	0.01236	0.00827
		2	1		1.0539	1.0331	1.0249	1.016	0.03404	0.01788	0.01336	0.00875
		0.5	1	2	0.97408	0.99194	0.99729	0.99536	0.02554	0.01503	0.01180	0.00804
		1	2		0.95913	0.9839	0.9918	0.99128	0.02427	0.01450	0.01147	0.00789
		2	2		0.98911	0.99977	1.0026	0.9994	0.02415	0.01471	0.01166	0.00795
		2	1		1.0205	1.0161	1.0136	1.0077	0.02771	0.01596	0.01236	0.00827

4-3: MSE of estimated inverted exponential parameter under squared error loss function.

method	parameters				Estimate for θ ($\hat{\theta}$)				MSE			
					Sample Size(n)				Sample Size(n)			
	θ	-	-	-	30	60	90	120	30	60	90	120
MLE	2.5				2.564	2.5445	2.5363	2.5231	0.21656	0.09685	0.08302	0.05358
Bayes	θ	v	a	b	$(P_1(\theta \backslash x))$ = Chi-square - Gamma distribution							
	2.5	1	0.5	1	2.2654	2.39	2.4322	2.4453	0.18318	0.09685	0.07357	0.04975
			1	2	2.1379	2.3164	2.3807	2.4061	0.229	0.10717	0.07679	0.05226
			2	2	2.208	2.3546	2.407	2.4261	0.18968	0.09701	0.07259	0.04963
			2	1	2.3786	2.4498	2.4728	2.4759	0.15598	0.09157	0.07204	0.04852
		2	0.5	1	2.3031	2.4099	2.4457	2.4555	0.17119	0.09428	0.07269	0.04913
			1	2	2.173	2.3355	2.3938	2.4161	0.20808	0.10172	0.07452	0.05084
			2	2	2.2431	2.3738	2.4201	2.4361	0.17378	0.09305	0.07102	0.04862
			2	1	2.4164	2.4697	2.4863	2.4861	0.15276	0.09142	0.07226	0.04852
Bayes	θ	v	λ	-	$(P_2(\theta \backslash x))$ = Chi-square - Erlang distribution							
	2.5	1	0.5		2.4745	2.5003	2.507	2.5016	0.16656	0.09671	0.07542	0.04998
			1		2.3786	2.4498	2.4728	2.4759	0.15598	0.09157	0.07204	0.04852
			2		2.208	2.3546	2.407	2.4261	0.18968	0.09701	0.07259	0.04963
		2	0.5		2.5138	2.5207	2.5207	2.5119	0.1714	0.09872	0.07663	0.05053
			1		2.4164	2.4697	2.4863	2.4861	0.15276	0.09142	0.07226	0.04852
			2		2.2431	2.3738	2.4201	2.4361	0.17378	0.09305	0.07102	0.04862
Bayes	θ	a	b	λ	$(P_3(\theta \backslash x))$ = Gamma - Erlang distribution							
	2.5	0.5	1	0.5	2.3786	2.4498	2.4728	2.4759	0.15598	0.09157	0.07204	0.04852
		1	2		2.2431	2.3738	2.4201	2.4361	0.17378	0.09305	0.07102	0.04862
		2	2		2.3132	2.4121	2.4201	2.456	0.14952	0.08736	0.07102	0.04719
		2	1		2.4919	2.5095	2.4464	2.5065	0.15509	0.09353	0.06892	0.04917
		0.5	1	1	2.2901	2.4013	2.4394	2.4507	0.16515	0.09188	0.07116	0.04843
		1	2		2.1656	2.3286	2.3884	2.4118	0.2053	0.10075	0.07375	0.05055
		2	2		2.2333	2.3662	2.4143	2.4316	0.17056	0.09160	0.06997	0.04815
		2	1		2.3991	2.4598	2.4794	2.481	0.14306	0.08780	0.07015	0.04751
		0.5	1	2	2.1318	2.3098	2.3754	2.4019	0.22618	0.10639	0.07615	0.05204
		1	2		2.026	2.2433	2.3273	2.3647	0.29615	0.12731	0.08503	0.05780
		2	2		2.0893	2.2795	2.3526	2.3841	0.24467	0.11204	0.07814	0.05359
		2	1		2.2333	2.3662	2.4143	2.4316	0.17056	0.09160	0.06997	0.04815

5. Discussion

In general, as we see in the tables (4-1) - (4-4) by using different estimation methods, we find the Mean Square Errors (MSE) was decreased when sample size increased in all cases. That means the estimation of the parameter θ $(\hat{\theta})$ get better for the large sample sizes. We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values for MSE the estimators θ under the squared error loss function with three different double priors.

In table (4-1) and in table (4-2),, when the true value of $\theta\,(\theta = 0.5, 1)$ in general , we obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE .We listed them when the double priors distribution for θ are

Chi-square - Gamma distribution with the values for the parameters v=1 and a=1, b=2.

- **Chi-square - Erlang distribution with the values for the parameters v=1 and $\lambda = 2$.**
- **Gamma - Erlang distribution with the values for the parameters a=1, b=2 and $\lambda = 2$.**

In table (4-3), when the true value of $\theta\,(\theta = 1.5)$ in general , we obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE .We listed them when the double priors distribution for θ are

- **Chi-square - Gamma distribution with the values for the parameters v=1 and a=2, b=2.**
- **Chi-square - Erlang distribution with the values for the parameters v=1 and $\lambda = 2$.**
- **Gamma - Erlang distribution with the values for the parameters a=2, b=2 and $\lambda = 2$.**

In table (4-4), when the true value of $\theta\,(\theta = 2.5)$ in general , we obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE.We listed them when the double priors distribution for θ are

- **Chi-square - Gamma distribution with the values for the parameters v=1 and a=2, b=1.**
- **Chi-square - Erlang distribution with the values for the parameters v=1 and $\lambda = 1$, also with v=2 and $\lambda = 1$.**
- **Gamma - Erlang distribution with the values for the parameters a=2, b=2 and $\lambda = 0.5$.**

See the summary of discussion for MSE in table (5-1) in Appendix-E.

6. Conclusion

When we compared the estimated values for θ $(\hat{\theta})$ for the parameter of the Exponential distribution by using the methods in this study .We find that Mean Square Errors (MSE) was decreased when sample size increased in all cases. And the MSE increased in all samples sizes (n) when the true value of θ increased .The best method is the bayes estimation according to the smallest values of MSE for all sample sizes (n) comparative to the estimated values by using MLE. We listed them when the double priors distribution for θ are

- Chi-square - Gamma distribution with the values for the parameters v=1 and a=1, b=2, when the true value of $\theta(\theta = 0.5)$.
- Gamma - Erlang distribution with the values for the parameters a=1, b=2 and $\lambda = 2$, when the true value of $\theta(\theta = 1)$.
- Gamma - Erlang distribution with the values for the parameters a=2, b=2 and $\lambda = 2$, when the true value of $\theta(\theta = 1.5)$.
- Gamma - Erlang distribution with the values for the parameters a=2, b=2 and $\lambda = 0.5$, when the true value of $\theta(\theta = 2.5)$.

See the summary of discussion for MSE in table (5-1) in Appendix-E.

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Appendix-A: The posterior distribution by using different double Priors

1- **The posterior distribution using Chi-square - Gamma distribution as double prior:**

To find The posterior distribution using Chi-squared distribution–Gamma distribution, we follow these steps:

When the prior distribution of θ take Chi-squared distribution, then the pdf is given by:

$$f_1(\theta)=\frac{1}{\Gamma(\frac{v}{2})}\,(\frac{1}{2})^{\frac{v}{2}}\;\theta^{\frac{v}{2}-1}\;\exp(-\frac{1}{2}\theta) \qquad \text{for} \quad \theta\geq 0\ ,\ v=1,2,\ \ldots v \qquad \ldots(A.1)$$

Again, when prior distribution of θ take gamma distribution, then the p.d.f. is given by :

$$f_2(\theta)=\frac{b^a}{\Gamma a}\,\theta^{a-1}\;\exp(-b\theta\ ) \qquad \text{for} \quad \theta\geq 0,\ \ a,b>0 \qquad \ldots(A.2)$$

We define the double prior for θ by combining these two priors as follows[1, 8]:

$$P_1(\theta)\ \alpha\ \ f_1(\theta)\ \ f_2(\theta) \qquad \ldots(A.3)$$

$$P_1(\theta)\ \alpha[\frac{1}{\Gamma(\frac{v}{2})}\,(\frac{1}{2})^{-\frac{v}{2}}\;\theta^{\frac{v}{2}-1}\exp(-\frac{1}{2}\theta)\ ][\frac{b^a}{\Gamma a}\quad \theta^{a-1}\ \exp(-b\theta\ )] \qquad \ldots(A.4)$$

$$P_1(\theta)\ \alpha[\frac{1}{\Gamma(\frac{v}{2})}\,(\frac{1}{2})^{-\frac{v}{2}}\,\frac{b^a}{\Gamma a}\ ]\,\theta^{\frac{v}{2}+a-2}\exp(-\theta(\frac{1}{2}+b)\ )$$

$$P_1(\theta)\ \alpha k\ \theta^{\frac{v}{2}+a-2}\ \exp(-\theta(\frac{1}{2}+b)\ ) \qquad \ldots(A.5)$$

Where $k=[\frac{1}{\Gamma(\frac{v}{2})}\,(\frac{1}{2})^{-\frac{v}{2}}\,\frac{b^a}{\Gamma a}\ ]$

Then the posterior distribution of θ for the given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is given by :

$$P(\theta\setminus t)=\frac{L(\underline{t}\setminus\theta)\ \ P(\theta)}{\int_\theta L(\underline{t}\setminus\theta)\ \ P(\theta)d\theta} \qquad \ldots(A.6)$$

Substituting the equation (4) and (A.5) in equation (A.6), we get:

$$P_i(\theta \backslash t)=\frac{\frac{1}{\prod_{i=1}^{n} t_i^2}\theta^n \exp(-\theta\sum_{i=1}^{n}\frac{1}{t_i})\ [k\ \theta^{\frac{v}{2}+a-2}\exp(-\theta(\frac{1}{2}+b))]}{\int_0^\infty \frac{1}{\prod_{i=1}^{n} t_i^2}\theta^n \exp(-\theta\sum_{i=1}^{n}\frac{1}{t_i})\ [k\ \theta^{\frac{v}{2}+a-2}\exp(-\theta(\frac{1}{2}+b))]d\theta} \quad \ldots (A.7)$$

$$P_i(\theta \backslash t)=\frac{\theta^{(n+\frac{v}{2}+a-1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b))}{\int_0^\infty \theta^{(n+\frac{v}{2}+a-1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b))\,d\theta} \quad \ldots (A.8)$$

By multiplying the integral in equation (A.8) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)}\right)\left(\frac{\Gamma(n+a+\frac{v}{2}-1)}{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2}-1)}}\right)$$

, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_i(\theta \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)\quad A(t,\theta)}\theta^{(n+\frac{v}{2}+a-1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)) \quad \ldots (A.9)$$

Where $A(t,\theta)$ equals to

$$A(t,\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)}\theta^{(n+\frac{v}{2}+a-1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b))\,d\theta=1.$$

Be the integral of the pdf of Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is

$$P_i(\theta \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)}\theta^{(n+\frac{v}{2}+a-1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)),\ a,n,b,v>0$$

$$\ldots (A.10)$$

It means that $P_1(\theta \setminus t)) \sim$ Gamma distribution with new parameters $(a_{(new)} = (n+a+\frac{v}{2}-1), b_{(new)} = (\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + b))$.

2- **The posterior distribution using Chi-square - Erlang distribution as double prior:**

To find The posterior distribution using Chi-squared distribution– Erlang distribution, we follow these steps:

When the prior distribution of θ take Chi-squared distribution, then the pdf is given by :

$$f_1(\theta) = \frac{1}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{-\frac{v}{2}} \theta^{\frac{v}{2}-1} \exp(-\frac{1}{2}\theta) \quad \text{for} \quad \theta \ge 0 , v = 1,2, \ ....v \quad ...(A.1)$$

Again, when prior distribution of θ take Erlang distribution, then the p.d.f is given by:

$$f_3(\theta) = \lambda^2 \ \theta \ \exp(-\lambda \ \theta) \quad \text{for} \quad \theta \ge 0 , \ \lambda > 0 \quad ... (A.11)$$

We define the double prior for θ by combining these two priors as follows [1, 8]:

$$P_2(\theta) \ \alpha \ f_1(\theta) \ f_3(\theta) \quad ...(A.12)$$

$$P_2(\theta) \ \alpha [\frac{1}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{-\frac{v}{2}} \theta^{\frac{v}{2}-1} \exp(-\frac{1}{2}\theta)][\lambda^2 \ \theta \ \exp(-\lambda \ \theta)] \quad ...(A.13)$$

$$P_2(\theta) \ \alpha [\frac{\lambda^2}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{-\frac{v}{2}}] \theta^{\frac{v}{2}+1-1} \exp(-\theta(\frac{1}{2}+\lambda))$$

$$P_2(\theta) \ \alpha \ k_2 \ \theta^{\frac{v}{2}+1-1} \exp(-\theta(\frac{1}{2}+\lambda)) \quad ... (A.14)$$

Where $k_2 = [\frac{\lambda^2}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{-\frac{v}{2}}]$

Then the posterior distribution of θ for the given the data $\underline{t} = (t_1, t_2, ..., t_n)$ is given by :

$$P(\theta \setminus t) = \frac{L(\underline{t} \setminus \theta) \ P(\theta)}{\int_\theta L(\underline{t} \setminus \theta) \ P(\theta) d\theta} \quad ...(A.6)$$

Substituting the equation (4) and (A.14) in equation (A.6), we get:

$$P_2(\theta \backslash t)=\frac{\frac{1}{\prod_{i=1}^{n} t_i^2}\,\theta^{n}\exp(-\theta\sum_{i=1}^{n}\frac{1}{t_i})\,[k_2\,\theta^{\frac{v}{2}+1-1}\exp(\theta(\frac{1}{2}+\lambda))\,]}{\int_0^\infty \frac{1}{\prod_{i=1}^{n} t_i^2}\,\theta^{n}\exp(-\theta\sum_{i=1}^{n}\frac{1}{t_i})\,[k_2\,\theta^{\frac{v}{2}+1-1}\exp(\theta(\frac{1}{2}+\lambda))\,]d\theta} \quad \ldots(A.15)$$

$$P_2(\theta \backslash t)=\frac{\theta^{(n+\frac{v}{2})+1-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda))}{\int_0^\infty \theta^{(n+\frac{v}{2})+1-1}\exp(\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda))\,d\theta} \quad \ldots(A.16)$$

By multiplying the integral in equation (A.16) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\right)\left(\frac{\Gamma(n+\frac{v}{2}+1)}{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}\right)$$

, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_2(\theta \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)\quad A1(t,\theta)}\theta^{(n+\frac{v}{2}+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)) \quad \ldots(A.17)$$

Where $A1(t,\theta)$ equals to

$$A1(t,\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\theta^{(n+\frac{v}{2}+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda))\,d\theta=1.$$

Be the integral of the pdf of Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is

$$P_2(\theta \backslash t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\theta^{(n+\frac{v}{2}+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)),\ n,v,\lambda>0 \quad \ldots(A.18)$$

It means that $P_2(\theta \setminus t)) \sim$ Gamma distribution with new parameters $(a_{(new)} = (n+\frac{v}{2}+1), b_{(new)} = (\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + \lambda))$.

3- **The posterior distribution using gamma - Erlang distribution as double prior:**

To find The posterior distribution using gamma distribution– Erlang distribution, we follow these steps:

When the prior distribution of θ take gamma distribution, then the pdf is given by:

$$f_1(\theta) = \frac{b^a}{\Gamma a} \theta^{a-1} \exp(-b\theta) \quad \text{for} \quad \theta \geq 0,\ a,b > 0 \qquad \ldots (A.2)$$

Again, when prior distribution of θ take Erlang distribution, then the pdf is given by:

$$f_2(\theta) = \lambda^2 \ \theta \ \exp(-\lambda\theta) \quad \text{for} \quad \theta \geq 0,\ \lambda > 0 \qquad \ldots (A.11)$$

We define the double prior for θ by combining these two priors as follows[1, 8]:

$$P_3(\theta) \ \alpha \ f_2(\theta) \ f_1(\theta) \qquad \ldots (A.19)$$

$$P_3(\theta) \ \alpha [\frac{b^a}{\Gamma a} \theta^{a-1} \exp(-b\theta)][\lambda^2 \ \theta \ \exp(-\lambda\theta)] \qquad \ldots (A.20)$$

$$P_3(\theta) \ \alpha [\frac{b^a}{\Gamma a} \lambda^2] \theta^a \exp(-\theta(b+\lambda))$$

$$P_3(\theta) \ \alpha \ k_2 \ \theta^a \exp(-\theta(b+\lambda)) \qquad \ldots (A.21)$$

Where $k_2 = [\frac{b^a}{\Gamma a} \lambda^2]$

Then the posterior distribution of θ for the given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is given by:

$$P(\theta \setminus t) = \frac{L(\underline{t} \setminus \theta) \ P(\theta)}{\int_\theta L(\underline{t} \setminus \theta) \ P(\theta) d\theta} \qquad \ldots (A.6)$$

Substituting the equation (4) and (A.21) in equation (A.6), we get:

$$P_3(\theta \setminus t) = \frac{\frac{1}{\prod_{i=1}^{n} t_i^2} \theta^n \exp(-\theta \sum_{i=1}^{n} \frac{1}{t_i}) [k_2 \ \theta^a \exp(-\theta(b+\lambda))]}{\int_0^\infty \frac{1}{\prod_{i=1}^{n} t_i^2} \theta^n \exp(-\theta \sum_{i=1}^{n} \frac{1}{t_i}) [k_2 \ \theta^a \exp(-\theta(b+\lambda))] d\theta} \qquad \ldots (A.22)$$

$$P_3(\theta \setminus t) = \frac{\theta^{(n+a)+1-1} \exp(-\theta(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda))}{\int_0^\infty \theta^{(n+a)+1-1} \exp(-\theta(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda)) d\theta} \qquad \ldots (A.23)$$

By multiplying the integral in equation (A.23) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)^{(n+a+1)}}{\Gamma(n+a+1)}\right)\left(\frac{\Gamma(n+a+1)}{(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)^{(n+a+1)}}\right)$$, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_3(\theta \setminus t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)^{(n+a+1)}}{\Gamma(n+a+1)\quad A2(t,\theta)}\theta^{(n+a+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)) \quad \ldots (A.24)$$

Where $A2(t,\theta)$ equals to

$$A2(t,\theta)=\int_0^{\infty}\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)^{(n+a+1)}}{\Gamma(n+a+1)}\theta^{(n+a+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda))\,d\theta=1$$

.Be the integral of the pdf of Gamma distribution . Then we get the posterior distribution of θ given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is

$$P_3(\theta \setminus t)=\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)^{(n+a+1)}}{\Gamma(n+a+1)}\theta^{(n+a+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)),\ a,b,n,\lambda>0 \quad \ldots (A.25)$$

It means that $P_3(\theta \setminus t))\sim$ Gamma distribution with new parameters ($a_{(new)}=(n+a+1), b_{(new)}=(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda)$).

Appendix-B

The following is the derivation of these estimators under the squared error loss function.

The squared error loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, the risk function is :

$$R(\hat{\theta} - \theta) = E[L_1(\hat{\theta} - \theta)] \qquad \text{...(B.1)}$$

$$R(\hat{\theta} - \theta) = \int_{\theta} L_1(\hat{\theta} - \theta)\ P(\theta \backslash x) d\theta$$

$$R(\hat{\theta} - \theta) = \int_{\theta} (\hat{\theta} - \theta)^2\ P(\theta \backslash x) d\theta \quad \Rightarrow R(\hat{\theta} - \theta) = \int_{\theta} (\hat{\theta}^2 - 2\theta\hat{\theta} + \theta^2)\ P(\theta \backslash x) d\theta$$

$$R(\hat{\theta} - \theta) = \hat{\theta}^2 \int_0^\infty P(\theta \backslash x) d\theta - 2\hat{\theta} \int_0^\infty \theta\, P(\theta \backslash x) d\theta + \int_0^\infty \theta^2\ P(\theta \backslash x) d\theta \Rightarrow$$

$$R(\hat{\theta} - \theta) = \hat{\theta}^2 - 2\hat{\theta} E(\theta \backslash x) + E(\theta^2 \backslash x) \qquad \text{... (B.2)}$$

Let $\dfrac{\partial}{\partial \hat{\theta}} R(\hat{\theta} - \theta) = 0$, we get Bayes estimator of θ denoted by $\hat{\theta}_{Bayes}$ for the above prior as follows

$$\hat{\theta}_{ss} = E(\theta \backslash x) = \int_0^\infty \theta\, P(\theta \backslash x) d\theta \qquad \text{... (B.3)}$$

1. Bayes estimation using Chi-square - Gamma distribution as double prior:

To obtain the Bayes' estimator under Chi-square - Gamma distribution as double prior. Substituting the equation (A.10) in equation (B.3), we get:

$$\hat{\theta}_{ss1} = \int_0^\infty \theta\, P_1(\theta \backslash x)\, d\theta$$

$$\hat{\theta}_{ss1} = \int_0^\infty \theta \frac{\left(\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + b\right)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)} \theta^{(n+\frac{v}{2}+a-1)-1} \exp\left(-\theta \left(\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + b\right)\right) d\theta \qquad \text{... (B.4)}$$

$$\hat{\theta}_{ss1} = \int_0^\infty \frac{\left(\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + b\right)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)} \theta^{(n+\frac{v}{2}+a)-1} \exp\left(-\theta \left(\sum_{i=1}^{n} \frac{1}{t_i} + \frac{1}{2} + b\right)\right) d\theta \qquad \text{... (B.5)}$$

By multiplying the integral in equation (B.5) by the quantity which equals to

$B_1 = \left(\dfrac{\Gamma(n+\frac{v}{2}+a)}{\Gamma(n+\frac{v}{2}+a)}\right)$, where $\Gamma(.)$ is a gamma function. Then, we have

$$\hat{\theta}_{SE1} = B1\int_0^\infty \frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2}-1)}}{\Gamma(n+a+\frac{v}{2}-1)}\theta^{(n+\frac{v}{2}+a)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b))d\theta \quad \ldots (B.6)$$

Then, we have

$$\hat{\theta}_{SE1} = \frac{\Gamma(n+a+\frac{v}{2})}{\Gamma(n+a+\frac{v}{2}-1)(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)}B2(t,\theta) \quad \ldots (B.7)$$

Where $B2(t,\theta)$ equals to

$$B2(t;\theta) = \int_0^\infty \frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)^{(n+a+\frac{v}{2})}}{\Gamma(n+a+\frac{v}{2})}\theta^{(n+\frac{v}{2}+a)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b))d\theta = 1.$$ Be the integral of the pdf of Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE1} = \frac{\Gamma(n+a+\frac{v}{2})}{\Gamma(n+a+\frac{v}{2}-1)(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+b)}, \quad n, b, a, v > 0 \quad \ldots (B.8)$$

2. Bayes estimation using Chi-square - Erlang distribution as double prior:

To obtain the Bayes' estimator under Chi-square - Erlang distribution as double prior. Substituting the equation (A.18) in equation (B.3), we get:

$$\hat{\theta}_{SE2} = \int_0^\infty \theta P_2(\theta \setminus x)d\theta$$

$$\hat{\theta}_{SE2} = \int_0^\infty \theta\frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\theta^{(n+\frac{v}{2}+1)-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda))d\theta \quad \ldots (B.9)$$

$$\hat{\theta}_{SE2} = \int_0^\infty \frac{(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\theta^{(n+\frac{v}{2}+1)+1-1}\exp(-\theta(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda))d\theta \quad \ldots (B.10)$$

By multiplying the integral in equation (B.10) by the quantity which equals to

$B3 = (\frac{\Gamma(n+\frac{v}{2}+2)}{\Gamma(n+\frac{v}{2}+2)})$, where $\Gamma(.)$ is a gamma function. Then, we have

$$\hat{\theta}_{se2} = B3\int_0^\infty \frac{\left(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda\right)^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\theta^{(n+\frac{v}{2}+1)}\exp\left(-\theta\left(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda\right)\right)d\theta \quad \ldots (B.11)$$

Then, we have

$$\hat{\theta}_{se2} = \frac{\Gamma(n+\frac{v}{2}+2)}{\Gamma(n+\frac{v}{2}+1)\left(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda\right)}B4(t,\theta) \quad \ldots (B.12)$$

Where B2(t,θ) equals to

$$B4(t,\theta) = \int_0^\infty \frac{\left(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda\right)^{(n+\frac{v}{2}+2)}}{\Gamma(n+\frac{v}{2}+2)}\theta^{(n+\frac{v}{2}+2)-1}\exp\left(-\theta\left(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda\right)\right)d\theta = 1.$$ Be

the integral of the pdf of Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{se2} = \frac{\Gamma(n+\frac{v}{2}+2)}{\Gamma(n+\frac{v}{2}+1)\left(\sum_{i=1}^{n}\frac{1}{t_i}+\frac{1}{2}+\lambda\right)} \qquad n,\ v,\lambda > 0 \quad \ldots (B.13)$$

3. Bayes estimation using gamma - Erlang distribution as double prior:

To obtain the Bayes' estimator under Chi-square - Erlang distribution as double prior. Substituting the equation (A.24) in equation (B.3), we get:

$$\hat{\theta}_{se3} = \int_0^\infty \theta P_3(\theta \setminus x)d\theta$$

$$\hat{\theta}_{se3} = \int_0^\infty \theta \frac{\left(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda\right)^{(n+a+1)}}{\Gamma(n+a+1)}\theta^{(n+a+1)-1}\exp\left(-\theta\left(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda\right)\right)d\theta \quad \ldots (B.14)$$

$$\hat{\theta}_{se3} = \int_0^\infty \frac{\left(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda\right)^{(n+a+1)}}{\Gamma(n+a+1)}\theta^{(n+a+1)+1-1}\exp\left(-\theta\left(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda\right)\right)d\theta \quad \ldots (B.15)$$

By multiplying the integral in equation (B.15) by the quantity which equals to $B5 = \left(\frac{\Gamma(n+a+2)}{\Gamma(n+a+2)}\right)$, where Γ(.)is a gamma function. Then, we have

$$\hat{\theta}_{se3} = B5\int_0^\infty \frac{\left(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda\right)^{(n+a+1)}}{\Gamma(n+a+1)}\theta^{(n+a+1)+1-1}\exp\left(-\theta\left(\sum_{i=1}^{n}\frac{1}{t_i}+b+\lambda\right)\right)d\theta \quad \ldots (B.16)$$

Then, we have

$$\hat{\theta}_{SE3} = B5\int_0^{\infty} \frac{\left(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda\right)^{(n+a+1)}}{\Gamma(n+a+1)} \theta^{(n+a+1)+1-1} \exp\left(-\theta\left(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda\right)\right) d\theta \quad \ldots (B.16)$$

Then, we have

$$\hat{\theta}_{SE3} = \frac{\Gamma(n+a+2)}{\Gamma(n+a+1)\left(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda\right)} B6(t,\theta) \qquad \ldots (B.17)$$

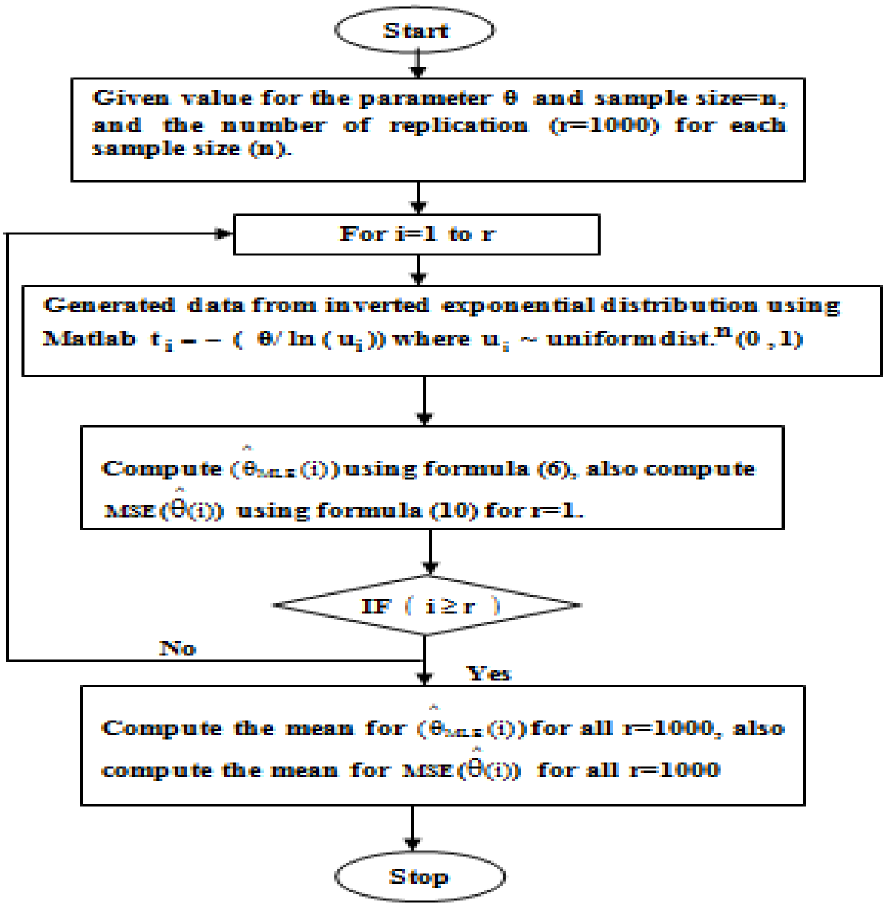
Where $B6(t,\theta)$ equals to

$$B6(t,\theta) = \int_0^{\infty} \frac{\left(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda\right)^{(n+a+2)}}{\Gamma(n+a+2)} \theta^{(n+a+2)-1} \exp\left(-\theta\left(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda\right)\right) d\theta = 1$$

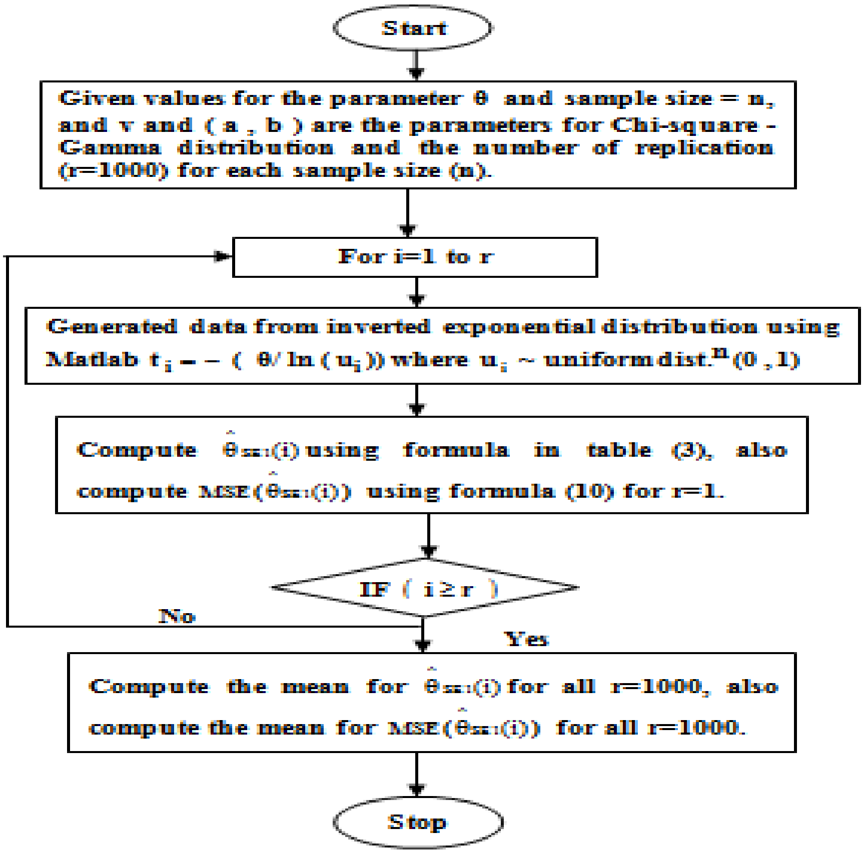
Be the integral of the pdf of Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE3} = \frac{\Gamma(n+a+2)}{\Gamma(n+a+1)\left(\sum_{i=1}^{n} \frac{1}{t_i} + b + \lambda\right)} \qquad n, b, a, \lambda > 0 \qquad \ldots (B.18)$$

Algorithm (1): To compute MLE for scale parameter ($\hat{\theta}$) with MSE.



Algorithm (2): To compute Bayes estimators ($\hat{\theta}_{SE1}$) using Chi-square - Gamma Distribution as double prior distribution for θ with MSE.



Note (1): we can reformulate the Algorithm (2) to compute Bayes estimators $\hat{\theta}_{SEk}, k = 2,3$ under using other distributions as double prior distribution for θ with MSE.

<u>Appendix-C</u>: The following is the programs algorithm.
<u>Appendix-E:</u> The summarized and tabulated discussions and conclusions.

Table 5-1: Best Estimation according to the smallest value for MSE.

method	parameters				Estimate for θ ($\hat{\theta}$) Sample Size(n)				MSE Sample Size(n)			
	θ	-	-	-	30	60	90	120	30	60	90	120
MLE	0.5				0.51281	0.50889	0.50505	0.50543	0.00866	0.00444	0.00295	0.00238
Bayes	θ	v	a	b	$(P_1(\theta \backslash x))$ = Chi-square - Gamma distribution							
	0.5	1	1	2	0.49936	0.50231	0.50075	0.5022	0.00736	0.00407	0.00279	0.00227
Bayes	θ	v	λ	-	$(P_2(\theta \backslash x))$ = Chi-square - Erlang distribution							
	0.5	1	2		0.51573	0.51061	0.50628	0.50637	0.0081	0.00432	0.00289	0.00235
Bayes	θ	a	b	λ	$(P_3(\theta \backslash x))$ = Gamma - Erlang distribution							
		1	2	2	0.511	0.50833	0.50482	0.50527	0.00743	0.00413	0.00281	0.00230
MLE	1				1.0256	1.0178	1.0145	1.0081	0.03464	0.01778	0.01328	0.00871
Bayes	θ	v	a	b	$(P_1(\theta \backslash x))$ = Chi-square - Gamma distribution							
	1	1	1	2	0.95837	0.98387	0.99186	0.99127	0.02661	0.01523	0.01186	0.00809
Bayes	θ	v	λ	-	$(P_2(\theta \backslash x))$ = Chi-square - Erlang distribution							
	1	1	2		0.9898	1.0001	1.0028	0.99949	0.02664	0.01547	0.01206	0.00815
Bayes	θ	a	b	λ	$(P_3(\theta \backslash x))$ = Gamma - Erlang distribution							
	1	1	2	2	0.95913	0.9839	0.9918	0.99128	0.02427	0.01450	0.01147	0.00789
MLE	1.5				1.5384	1.519	1.5182	1.5185	0.07796	0.03899	0.02553	0.02137
Bayes	θ	v	a	b	$(P_1(\theta \backslash x))$ = Chi-square - Gamma distribution							
	1.5	1	2	2	1.4272	1.4629	1.4804	1.49	0.05615	0.03290	0.02240	0.01908
Bayes	θ	v	λ	-	$(P_2(\theta \backslash x))$ = Chi-square - Erlang distribution							
	1.5	1	2		1.4272	1.4629	1.4804	1.49	0.05615	0.03290	0.02240	0.01908
Bayes	θ	a	b	λ	$(P_3(\theta \backslash x))$ = Gamma - Erlang distribution							
	1.5	2	2	2	1.3979	1.4463	1.4687	1.4809	0.05290	0.03153	0.02162	0.01843
MLE	2.5				2.564	2.5445	2.5363	2.5231	0.21656	0.09685	0.08302	0.05358
Bayes	θ	v	a	b	$(P_1(\theta \backslash x))$ = Chi-square - Gamma distribution							
	2.5	1	2	1	2.3786	2.4498	2.4728	2.4759	0.15598	0.09157	0.07204	0.04852
		2	2	1	2.4164	2.4697	2.4863	2.4861	0.15276	0.09142	0.07226	0.04852
Bayes	θ	v	λ	-	$(P_2(\theta \backslash x))$ = Chi-square - Erlang distribution							
	2.5	1	1		2.3786	2.4498	2.4728	2.4759	0.15598	0.09157	0.07204	0.04852
	2.5	2	1		2.4164	2.4697	2.4863	2.4861	0.15276	0.09142	0.07226	0.04852
Bayes	θ	a	b	λ	$(P_3(\theta \backslash x))$ = Gamma - Erlang distribution							
	2.5	2	2	0.5	2.3132	2.4121	2.4201	2.456	0.14952	0.08736	0.07102	0.04719

مقدرات بيز لمعلمة توزيع Inverted Exponential باستعمال دوال معلوماتية مضاعفه

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الخلاصة

في هذا البحث ، نقدم مقارنة لدوال معلوماتية مضاعفة التي تفترض لمعلمة توزيع الاسي المقلوب، لتقدير معلمة توزيع الاسي المقلوب باستعمال تقديربيز.استخدمنا نوعين مختلفين من المعلومات في طريقة تقدير بيز، اختيرت نوعين مختلفين من الدوال الاولية لمعلمة توزيع الاسي المقلوب .هنا افترضنا توزيع مربع كاي – كاما ، توزيع مربع كاي- ارلنك وتوزيع كاما - ارلنك كدوال معلوماتية مضاعفه، نتائج الاشتقاقات لتلك المقدرات باستعمال دالة الخسارة التربيعية مع ثلاثة دوال معلوماتية مضاعفه. كذلك استعملنا طريقة الامكان الاعظم (MLE) لتقدير معلمة توزيع الاسي المقلوب . استعمل اسلوب المحاكاة في مقارنة اداء كل مقدر، بافتراض عدة حالات لمعلمة توزيع الاسي المقلوب استعملت لتوليد البيانات ولاحجام مختلفة من العينات (صغيرة ، متوسطة ، كبيرة). وقد اظهرت نتائج المحاكاة بان طريقة بيز الافضل وفقا لمقياس اقل قيمة متوسط مربع الاخطاء (MSE) ، مقارنة بطريقة الإمكان الأعظم (MLE) .وفقا للنتائج المستحصلة ، نرى بانه عندما تكون الدالة المعلوماتية المضاعفة هي توزيع كاما - ارلنك عند قيم معينة لمعلمات الدالة المعلوماتية المضاعفة ، اعطى نتائج افضل وفقا لاقل قيمة لـمتوسط مربعات الخطاء (MSE) مقارنة بنفس القيم المستحصلة بطريقة الامكان الاعظم (MLE) ،عندما تكون القيمة الحقيقة المفترضة لـ$\theta = 1, 1.5, 2.5$ ولكل حجوم العينات (n).

المصطلحات الرئيسة للبحث/ توزيع الاسي المقلوب ،طريقة الإمكان الأعظم، ،طريقة بيز ،الدوال المعلوماتية المضاعفة : توزيع مربع كاي- كاما ، توزيع مربع كاي- ارلنك ، توزيع كاما - ارلنك ، دالة الخسارة التربيعية .