Using simulation to estimate parameters and reliability function for extreme value distribution

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Abstract

This study includes Estimating scale parameter (θ), location parameter (γ) and reliability function $R(t)$ for Extreme Value (EXV) distribution by two methods, namely: -
- Maximum Likelihood Method (MLE).
- Probability Weighted Moments Method (PWM).
Used simulations to generate the required samples to estimate the parameters and reliability function of different sizes(n=10,25,50,100) , and give real values for the parameters are $(\theta = 0.5,1,1.5)$ and $(\gamma = 3)$, replicate the simulation experiments (R_P=1000), and estimated the function reliability at two times (T=2,5), adopted mean square error for comparison between the results of estimators , and summarized all the results in tables especially prepared for this purpose, An analysis of the results in the tables it becomes that:-
Estimators of (MLE) better than estimators of (PWM) in most of the cases studied in this research.

المستخلص

يتضمن هذا البحث تقدير معلمتي القياس والموقع (θ, γ) والدالة المعولية $R(t)$ لتوزيع القيمة المتطرفة (EXV) بواسطة طريقتين وهما:-
- طريقة الامكان الاعظم (MLE) .
- طريقة العزوم الاحتمالية المرجحة (PWM) .
حيث تم توليد بيانات عند حجوم عينات مختلفة (n=10,25,50,100)، واعطيت قيم افتراضية اولية للمعلمتين $(\theta = 0.5,1,1.5)$ و$(\gamma = 3)$، وكررت تجارب المحاكاة (R_P=1000) ، وقدرت دالة المعولية عند الزمنين (T=2,5) ،ومن ثم اعتمد المقياس الاحصائي متوسط مربع الخطأ Mean Square Error ((MSE)) للمقارنة بين نتائج المقدرات،ولخصت جميع النتائج في جداول خاصة اعدت لهذا الغرض، ولوحظ ان مقدرات (MLE) كانت افضل من(PWM) لمعظم الحالات المدروسة.

<u>Keywords</u>: extreme value distribution, Maximum Likelihood method (MLE), Probability Weighted Moments method (PWM), reliability function, means squared error.

1. Introduction

The extreme value distribution has been used widely in the field of reliability theory, and is considered one of the statistical distributions and the continuing failure models one common task[1].
If the random variable T follows the extreme value distribution with two parameters (θ, γ), then it's probability density function (p.d.f) and also cumulative distribution function (C.D.F), are as the follows:[4]

$$f_T(t;\theta,\gamma) = \frac{1}{\theta}\exp\left(\frac{t-\gamma}{\theta}\right)\exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right) \quad \ldots (1)$$

$$F_T(t) = 1 - \exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right) \quad \ldots (2)$$

Where
t is Value of the random variable, $-\infty < t < \infty$
θ **Scale parameter,** $\theta > 0$
γ **Location parameter,** $\gamma \geq 0$

Reliability (R) is defined as the probability that any system will perform its required function for a given period of time t when used under stated operating conditions. In a mathematical language that yields:

$$R(t) = \Pr(T > t)$$

The failure law is the probability that a failure occurs before time t, it is the cumulative distribution function (C.D.F) of the failure distribution:

$$F(t) = \Pr(T \leq t) = 1 - R(t)$$

Finally, the reliability function $R(t)$ is the conditional probability of failure in the time interval [0, t) given that the system has survived to time t:[1]

$$R(t) = \exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right) \quad \ldots (3)$$

2. Methods of Estimation

In this section, we display two methods for estimating the two parameters and reliability function of extreme value distribution, which are (MLE) and (PWM) .

1-2 Probability Weighted Moments method (PWM)

The general formula of the (PWM) method was proposed by (Green Wood) (1979) [8], Although it is an independent method to find estimator for the parameters it is also one of the methods in which can be used it's estimator a preliminary estimation for other methods, This has been the use of the estimator of this method as initial estimates of the (MLE) method. [5]

We can depend formula of (PWM) version of the order q, r, s (q, r and s are non-negative integers) of the random variable T which represents the distribution function F(t) as follow:[2]

$$M_{q,r,s} = E\left[[t(F)]^q [F(t)]^r [1-F(t)]^s\right] \quad \ldots \quad (4)$$

$$= \int_0^\infty [t(F)]^q [F(t)]^r [1-F(t)]^s dF$$

t(F) is Inverse distribution function F(t) , when we offset formula (2) in the formula (4) we get the following formula [2]

$$M_{q,r,s} = E\left[\begin{array}{l} [\gamma + \theta \ln(-\ln(1-F(t)))]^q \left[1-\exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right)\right]^r \\ \left[\exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right)\right]^s \end{array}\right] \ldots (5)$$

$$M_{1,0,S} = E\left[[\gamma + \theta \ln(-\ln(1-F(t)))]\left[\exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right)\right]^S\right]$$

$$= E\left[t(F)[1-F(t)]^S\right] \quad \ldots \quad (6)$$

When we find the expected value of the formula (6) they become as the follow

$$M_{1,0,S} = \exp(\gamma)(1+S)^{S}\Gamma(\delta)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{\binom{n-i}{S}}{\binom{n-1}{S}}\exp(t_i) \qquad \dots \ (7)$$

So that $\delta = 1+\theta$

When the compensation values s = 0, 1, 2 in the formula (7), respectively, we get the following formulas

$$M_{1,0,0} = \frac{1}{n}\sum_{i=1}^{n}\exp(t_i)$$

$$M_{1,0,1} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{n-i}{n-1}\right)\exp(t_i)$$

$$M_{1,0,2} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{(n-i)(n-i-1)}{(n-1)(n-2)}\right)\exp(t_i)$$

In the following two formulas we can estimate scale parameter θ **and location parameter** γ **respectively**

$$\hat{\theta}_{pwm} = \hat{\delta} - 1 \qquad \dots \ (8)$$

$$\hat{\gamma}_{pwm} = Log\left(\frac{M_{1,0,0}}{\Gamma(\hat{\delta})}\right) \qquad \dots \ (9)$$

So that

$$\hat{\delta} = \frac{Log\left(\frac{M_{1,0,0}}{M_{1,0,1}}\right)}{Log(2)}$$

So the (PWM) estimate of the reliability function $R(t)$ is

$$\widehat{R}_{PWM}(t)=\exp\left(-\exp\left(\frac{t-\widehat{\gamma}_{PWM}}{\widehat{\theta}_{PWM}}\right)\right) \qquad \dots \ (10)$$

2-2 Maximum Likelihood method (MLE)

The manner (MLE) of the most common method, which can be obtained by which the destinies of the values of efficient when compared with other estimation methods, In order to find estimates of the values of two parameters of the extreme value distribution function, the maximum likelihood (Lf) of the formula (1) are as follows:[7]

$$Lf\left(t;\theta,\gamma\right)=\prod_{i=1}^{n}\left[\frac{1}{\theta}\exp\left(\frac{t-\gamma}{\theta}\right)\exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right)\right]$$

$$=\frac{1}{\theta^{n}}\exp\left(\sum_{i=1}^{n}\frac{t_i-\gamma}{\theta}\right)\exp\left(\sum_{i=1}^{n}-\exp\left(\frac{t_i-\gamma}{\theta}\right)\right) \qquad \dots \ (11)$$

Taking the natural logarithm of the formula (11) as they become as follow

$$LLf\left(t;\theta,\gamma\right)=-nLn\theta+\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\theta}\right)-\sum_{i=1}^{n}\exp\left(\frac{t_i-\gamma}{\theta}\right) \qquad \dots \ (12)$$

The application of the method (MLE) should be taking first and second partial derivatives of formula (12) with respect to scale and location parameters as follows:

$$\frac{\partial LLf}{\partial\gamma}=-\frac{n}{\theta}+\frac{1}{\theta}\sum_{i=1}^{n}\exp\left(\frac{t_i-\gamma}{\theta}\right)$$

$$\frac{\partial^2 LLf}{\partial\gamma^2}=-\frac{1}{\theta^2}\sum_{i=1}^{n}\exp\left(\frac{t_i-\gamma}{\theta}\right)$$

$$\frac{\partial^2 LLf}{\partial\gamma\partial\theta}=\frac{n}{\theta^2}-\frac{1}{\theta^2}\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\theta}\right)\exp\left(\frac{t_i-\gamma}{\theta}\right)-\frac{1}{\theta^2}\sum_{i=1}^{n}\exp\left(\frac{t_i-\gamma}{\theta}\right)$$

$$\frac{\partial LLf}{\partial\theta}=-\frac{n}{\theta}-\frac{1}{\theta}\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\theta}\right)+\frac{1}{\theta}\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\theta}\right)\exp\left(\frac{t_i-\gamma}{\theta}\right)$$

$$\frac{\partial^2 LLf}{\partial\theta^2}=\frac{n}{\theta^2}+\frac{2}{\theta^2}\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\theta}\right)-\frac{1}{\theta^2}\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\theta}\right)^2\exp\left(\frac{t_i-\gamma}{\theta}\right)-\frac{2}{\theta^2}\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\theta}\right)\exp\left(\frac{t_i-\gamma}{\theta}\right)$$

by solving the formulas below, we get the values of $\delta\gamma_i$ **and** $\delta\theta_i$

$$-\frac{\partial LLf}{\partial\gamma}=\frac{\partial^2 LLf}{\partial\gamma^2}\delta\gamma_i+\frac{\partial^2 LLf}{\partial\gamma\partial\theta}\delta\theta_i$$

$$-\frac{\partial LLf}{\partial\theta}=\frac{\partial^2 LLf}{\partial\gamma\partial\theta}\delta\gamma_i+\frac{\partial^2 LLf}{\partial\theta^2}\delta\theta_i$$

And the estimated values for two parameters of the extreme value distribution get it of the two formulas below

$$\hat{\theta}^{(i+1)}{}_{MLE}=\hat{\theta}_i+\delta\theta_i \quad \text{... (13)}$$

$$\hat{\gamma}^{(i+1)}{}_{MLE}=\hat{\gamma}_i+\delta\gamma_i \quad \text{... (14)}$$

So the (MLE) estimate of the reliability function $R(t)$ **is**

$$\widehat{R}_{MLE}(t)=\exp\left(-\exp\left(\frac{t-\widehat{\gamma}_{MLE}}{\widehat{\theta}_{MLE}}\right)\right) \quad \text{... (15)}$$

3- Simulation

We obtained, in the above Sections, white and modify white estimates for the scale and location parameters θ **and** γ**, reliability** $R(t)$ **function of the (EXV) distribution.**

In order to assess the statistical performances of these estimates, a simulation study is conducted. The mean square errors (MSE's) using generated random samples of different sizes are computed for each estimator. The random samples are generated as follows:

1. Given values of the scale and threshold parameters $(\theta=0.5,1,1.5)$ **and** $(\gamma=3)$.

2. Using θ and γ, obtained in step (1), and generate random samples of different sizes: n=10,25,50 and 100 ,replicate size: N=1000 from the (EXV) distribution by using the following:
From the formula (2) we obtained

$$\exp\left(-\exp\left(\frac{t-\gamma}{\theta}\right)\right)=1-F(t)$$

so

$$\frac{t-\gamma}{\theta}=Ln(-Ln(1-F(t)))$$

If $U=F(t)$ where U is continuous random on [0,1] then we obtained a random samples from the following

$$t=\gamma+\theta(Ln(-Ln(1-U)))$$

3. The white and modify white estimators of the parameters θ and γ , $(\hat{\theta}_{PWM},\hat{\theta}_{MLE})$ and $(\hat{\gamma}_{PWM},\hat{\gamma}_{MLE})$, are obtained by solving the equations (8),(13),(9)and(14) respectively. The estimators $\hat{R}_{PWM}(t)$ and $\hat{R}_{MLE}(t)$ of the function $R(t)$ are computed at values T=2,5 from (10) and (15) respectively .

4. The above steps are repeated 1000 times and the mean square errors (MSE) are computed for different sample sizes (n) and run sizes N by using the following:

$$MSE(\hat{\theta})=\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i-\theta)^2$$

4- The Results

After applying both the method (MLE) and (PWM) on the data generated to estimate the parameters, the results of extreme value estimate and the values of MSE and the values of Bias to these estimates, as shown in the following tables

1-4 Results of θ

Table (1)
Values of θ estimators

θ \ Est.	$\hat{\theta}_{PWM}$	$\hat{\theta}_{MLE}$	n
0.5	0.50846	0.45792	10
	0.50109	0.48195	25
	0.50281	0.49331	50
	0.50178	0.49722	100
1	1.02004	0.91415	10
	1.00393	0.96390	25
	1.00731	0.98664	50
	1.00423	0.99443	100
1.5	1.48746	1.36128	10
	1.49988	1.44514	25
	1.50966	1.47996	50
	1.50537	1.49166	100

Table (2)
Values of (MSE) for θ estimators

θ \ Est.	$\hat{\theta}_{PWM}$	$\hat{\theta}_{MLE}$	n
0.5	0.01830	0.01589	10
	0.00713	0.00663	25
	0.00346	0.00321	50
	0.00180	0.00166	100
1	0.07820	0.06300	10
	0.03078	0.02654	25
	0.01497	0.01282	50
	0.00767	0.00665	100
1.5	0.17019	0.14339	10
	0.08159	0.05944	25
	0.04133	0.02885	50
	0.02059	0.01497	100

2-5 Results of γ

Table (3)
Values of γ estimators

θ \ Est.	$\hat{\gamma}_{PWM}$	$\hat{\gamma}_{MLE}$	n
0.5	2.98173	2.98560	10
	2.98837	2.98966	25
	2.99098	2.99177	50
	2.99520	2.99562	100
1	2.92239	2.97225	10
	2.96079	2.97932	25
	2.97391	2.98354	50
	2.98657	2.99124	100
1.5	2.85473	2.96649	10
	2.92012	2.97002	25
	2.94883	2.97532	50
	2.97407	2.98687	100

Table (4)
Values of (MSE) for γ estimators

θ \ Est.	$\hat{\gamma}_{PWM}$	$\hat{\gamma}_{MLE}$	n
0.5	0.02836	0.02771	10
	0.01154	0.01141	25
	0.00559	0.00557	50
	0.00328	0.00284	100
1	0.12445	0.11057	10
	0.04843	0.04620	25
	0.02310	0.02230	50
	0.01172	0.01137	100
1.5	0. 28997	0.24178	10
	0.11755	0.10207	25
	0.05625	0.05017	50
	0.02758	0.02559	100

3-5 Results of Reliability

Table (5)
Values of Reliability estimators

T	θ	$\hat{R}_{PWM}$	$\hat{R}_{MLE}$	n
2	0.5	0.86024	0.88210	10
		0.86782	0.87693	25
		0.86891	0.87350	50
		0.87085	0.87309	100
	1	0.67330	0.70897	10
		0.68384	0.69776	25
		0.68502	0.69226	50
		0.68842	0.69190	100
	1.5	0.57968	0.61571	10
		0.58764	0.60278	25
		0.58950	0.59757	50
		0.59390	0.59777	100
5	0.5	1.64022×10^{-6}	1.78997×10^{-7}	10
		1.12533×10^{-9}	8.74046×10^{-11}	25
		1.00593×10^{-11}	1.72754×10^{-12}	50
		5.71287×10^{-14}	1.49569×10^{-14}	100
	1	0.00388	0.00242	10
		0.00181	0.00133	25
		0.00127	0.00103	50
		0.00095	0.00084	100
	1.5	0.02556	0.02427	10
		0.02324	0.02331	25
		0.02314	0.02244	50
		0.02280	0.02249	100

Table (6)
Values of (MSE) for Reliability estimators

T / θ	Est θ	$\hat{R}_{PWM}$	$\hat{R}_{MLE}$	n
2	0.5	0.00881	0.00597	10
		0.00304	0.00318	25
		0.00135	0.00128	50
		0.00074	0.00071	100
	1	0.05292	0.06700	10
		0.00673	0.00696	25
		0.00307	0.00289	50
		0.00170	0.00165	100
	1.5	0.02354	0.02890	10
		0.01546	0.01208	25
		0.00380	0.00326	50
		0.00203	0.00185	100
5	0.5	5.36128×10^{-10}	1.23990×10^{-11}	10
		2.02797×10^{-16}	1.23523×10^{-18}	25
		3.05654×10^{-20}	1.20923×10^{-21}	50
		5.97776×10^{-25}	7.196558×10^{-26}	100
	1	0.00008	0.00004	10
		0.00001	0.000006	25
		0.000003	0.000002	50
		0.000001	0	100
	1.5	0.00089	0.00098	10
		0.00039	0.00038	25
		0.00022	0.00020	50
		0.00012	0.00011	100

6- Analysis of the results

a. **When we analyze the results of the parameter θ in table (1) has become quite clear that the probability weighted moments method is better than maximum likelihood method, but the results in table (2) shows the Maximum Likelihood method better than probability weighted moments method.**
b. **When we analyze the results of the parameter γ in tables (3) and (4) has become quite clear that the maximum likelihood method better than probability weighted moments method in all cases.**
c. **When we analyze the results of the reliability function $R(t)$ in the table (5) shows that (PWM) method is better than (MLE) when T=2, but when T=5 the (MLE) is better than (PWM).**

The results in table (6) shows the (MLE) method better than (PWM) method in all cases except in cases (n=25,T=2,θ=0.5) , (n=10,T=2,θ=1) , (n=25,T=2,θ=1) and (n=10,T=2,θ=1.5) .

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