مقدرات الاختبار الأولي المقلصة ذات المرحلة الواحدة لمعلمة القياس للتوزيع الأسي

أ. م.عباس نجم سلمان   م. م. بيداء عطية خلف   م. م.سيران حمزة

جامعة بغداد- كلية التربية- ابن الهيثم - قسم الرياضيات

الخلاصة

استخدم اسلوب الاختبار الأولي المتقلص ذو المرحلة الواحدة في تقدير معلمة القياس (θ) للتوزيع الأسي عند توافر معلومات أولية عن θ بشكل قيمة ابتدائية θ_0. حيث اقترح مقدر الاختبار الأولي $\tilde{\theta}$ في تقدير θ بالاعتماد على نتيجة اختبار الفرضية $H_0: \theta = \theta_0$ ضد الفرضية $H_A: \theta \neq \theta_0$ بمستوى معنوية α. ففي حالة قبول الفرضية H_0 سيكون المقدر المقترح $\tilde{\theta}$ هو المقدر المتقلص $\tilde{\theta} = \psi_1(\hat{\theta})(\hat{\theta} - \theta_0) + \theta_0$ حيث ان $\psi_1(\hat{\theta})$ هي عامل او دالة تقلص ربما تكون دالة الاختبار عند اختبار H_0 أو ربما تكون ثابتة حيث ان، $0 \le \psi_1(\hat{\theta}) \le 1$ و $\hat{\theta}$ هو مقدر غير متحيز. وفي حالة رفض الفرضية H_0 فيكون المقدر $\tilde{\theta}$ هو المقدر المتقلص $\tilde{\theta} = \psi_2(\hat{\theta})(\hat{\theta} - \theta_0) + \theta_0$، حيث ان $0 \le \psi_2(\hat{\theta}) \le 1$ وهي دالة تقلص موزونة أيضاً. تم اختيار دوال تقلص موزونة $\psi_i(\hat{\theta})$ ، (i =1,2) بشكل مناسب ومن ثم تم اشتقاق معادلات التحيز ومعدل مربعات الخطأ للمقدرات المقترحة. قورنت المقدرات المقترحة مع المقدر الكلاسيكي ($\hat{\theta}$) وبعض المقدرات المقترحة في الدراسات السابقة لبيان كفاءتها.

Abstract

A preliminary test single stage Shrinkage (PTSSS) techniques is used for estimation the scale parameter θ of an exponential distribution when a prior knowledge about θ is a available in the form of initial estimate θ_0 of θ. It is proposed to estimate θ by a testimator $\tilde{\theta}$ that is based upon the result of a test of the hypothesis $H_0: \theta = \theta_0$ against the hypothesis $H_A: \theta \neq \theta_0$ with level of significance α. If H_0 is accepted we take $\tilde{\theta} = \psi_1(\hat{\theta})(\hat{\theta} - \theta_0) + \theta_0$, where the weighting factor $\psi_1(\hat{\theta})$ is a function of the test statistic for testing H_0 or may be constant such that $0 \le \psi_1(\hat{\theta}) \le 1$. However if H_0 is rejected we take $\tilde{\theta} = \psi_2(\hat{\theta})(\hat{\theta} - \theta_0) + \theta_0$, where $0 \le \psi_2(\hat{\theta}) \le 1$ and $\hat{\theta}$ is the classical estimator of θ (MLE). Choosing the weighting factor $\psi_i(\hat{\theta})$, (i =1,2) appropriately, an expression for the mean squared error (MSE) and Bias of $\tilde{\theta}$ are derived and comparisons are made with classical estimator ($\hat{\theta}$) in the sense of efficiency and last earlier studies.

2. Introduction

Exponential distribution is on of the most useful and widly exploited model, Epstein [1] remarks that the exponential distribution plays as important a role in life experiments as the part played by the normal distribution in agricultural experiments.

The most widely model is one parameter exponential distribution with probability density function

$$f(t) = \frac{1}{\theta}\exp(\frac{-t}{\theta}),$$

where $t \geq 0$, $\theta > 0$ is the average or the mean life or mean time to failure (MTTF) and it is also acts as scale parameter.

In this paper we consider Shrinkage technique which introduced by Thompson [2] as follows

$$\tilde{\theta} = \varphi(\hat{\theta})(\hat{\theta} - \theta_0) + \theta_0 \qquad \ldots(1)$$

where θ_0 is a prior information about (θ) and $0 \leq \varphi(\hat{\theta}) \leq 1$, represents a Shrinkage weight factor specifying the degree of belief in θ_0. We used the form (1) above to estimate the scale parameter θ of an exponential distribution.

In case $\varphi(\hat{\theta})$ is chosen as follows

$$\varphi(\hat{\theta}) = \begin{cases} \psi_1(\hat{\theta}) & ,\text{if } \hat{\theta} \in R \\ \psi_2(\hat{\theta}) & ,\text{if } \hat{\theta} \notin R \end{cases} \qquad \ldots(2)$$

where R is the preliminary test region of acceptance of size α for testing the hypothesis $H_0 : \theta = \theta_0$ against the hypothesis $H_A : \theta \neq \theta_0$ using the test statistic $T(\hat{\theta}/\theta) = \frac{2n\hat{\theta}}{\theta_0}$ and $\hat{\theta}$ is the classical estimater of θ (MLE), then the estimator which is defined in (1) will be written as below

$$\tilde{\theta} = \begin{cases} \psi_1(\hat{\theta} - \theta_0) + \theta_0 & ,\text{if } \hat{\theta} \in R \\ \psi_2(\hat{\theta} - \theta_0) + \theta_0 & ,\text{if } \hat{\theta} \notin R \end{cases}$$

…(3)

where $\psi_i(\hat{\theta})$, i =1,2, $0 \leq \psi_i(\hat{\theta}) \leq 1$ represents as Shrinkage weight factors which may be a functions of $\hat{\theta}$ or may be constants.

The resulting estimator (3) is known as preliminary test single stage Shrinkage estimator (PTSSSE).

Several authors had studied the estimator defined in (3) for special distribution and for different parameters also for complete and censored samples as well as for estimate the parameters of linear regression model. For example see [3], [4], [5], [6] and [7].

3. Preliminary Test Single Stage Shrinkage Estimators (PTSSSE)

Using the form (3), we proposed two preliminary single stage Shrinkage estimators $\tilde{\theta}_1$ and $\tilde{\theta}_2$ for estimate the scale parameter θ of an exponent distribution when a prior information (θ_0) available about the actual value (θ).

$$\tilde{\theta}_1 = \begin{cases} \theta_0 & ,\text{if } \hat{\theta} \in R \\ \hat{\theta} & ,\text{if } \hat{\theta} \notin R \end{cases} \quad \ldots(4)$$

i.e. $\psi_1(\hat{\theta})$ = 0 and $\psi_2(\hat{\theta})$ = 1 in (3).

and

$$\tilde{\theta}_2 = \begin{cases} \theta_0 & ,\text{if } \hat{\theta} \in R \\ k(\hat{\theta} - \theta_0) + \theta_0 & ,\text{if } \hat{\theta} \notin R, \end{cases}$$

…(5) i.e. $\psi_1(\hat{\theta})$ = 0 and $\psi_2(\hat{\theta})$ = k in (3)

where R defined in (2) such that

$$R = [\frac{\theta_0}{2n} X^2_{1-\alpha/2,2n}, \frac{\theta_0}{2n} X^2_{\alpha/2,2n}]$$

…(6) Assume that R = [*a*,*b*], *a* < *b*

i.e. $a = \frac{\theta_0}{2n} X^2_{1-\alpha/2,2n}, b = \frac{\theta_0}{2n} X^2_{\alpha/2,2n}$ **…(7)**

where $X^2_{1-\alpha/2,2n}$ and $X^2_{\alpha/2,2n}$ are respectively the lower and upper 100(α/2) percentile point of chi-square distribution with (2n) degree of freedom.

The expressions for Bias [B(·)] of the estimators $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are respectively as follows

$$B(\tilde{\theta} | \theta, R) = E(\tilde{\theta}_1) - \theta$$

$$= \int_R [\theta_0 - \theta] f(\hat{\theta}) d\hat{\theta} + \int_{\bar{R}} [\hat{\theta} - \theta] f(\hat{\theta}) d\hat{\theta}$$

where $\bar{R}$ is the complement region of R in real space and $f(\hat{\theta})$ is a p.d.f. of $\hat{\theta}$ which has the following forms

$$f(\hat{\theta}) = \frac{[\hat{\theta}]^{n-1} \exp[-n\hat{\theta}/\theta]}{\Gamma(n)(\theta/n)^n} \quad \text{for } 0 < \hat{\theta} < \infty$$

…(8) we conclude,

$$B(\tilde{\theta}_1 | \theta, R) = \theta\{\lambda j_0(a^*, b^*) - j_1(a^*, b^*)\}$$

…(9) where $j_\ell(a^*, b^*) = \frac{1}{n^\ell \Gamma(n)} \int_{a^*}^{b^*} y^\ell y^{n-1} e^{-y} dy,$ **for** ℓ **=0,1,2,…**

…(10) $\lambda = \theta_0 / \theta$, *a = λ $X^2_{1-\alpha/2,2n}$ and *b** = λ $X^2_{\alpha/2,2n}$**

…(11) and,

$$B(\tilde{\theta}_2 \mid \theta, R) = E(\tilde{\theta}_2 - \theta)$$

$$= \theta\{(\lambda-1)(j_0(a^*,b^*)+1) - k(j_1(a^*,b^*) + j_0(a^*,b^*)\lambda)\} \quad ...(12)$$

The expression for mean square error (MSE) of $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are respectively given as below:

$$MSE(\tilde{\theta}_1 \mid \theta, R) = E(\tilde{\theta}_1 - \theta)^2$$

$$= \theta^2 \left\{ (\lambda-1)^2 j_0(a^*,b^*) + \frac{1}{n} - [j_2(a^*,b^*) - 2j_1(a^*,b^*) + j_0(a^*,b^*)] \right\}$$

…(13) and

$$MSE(\tilde{\theta}_2 \mid \theta, R) = \theta^2 \{ k^2/n + (\lambda-1)^2 (k^2+2) - 2k(\lambda-1)[\lambda(1 - j_0(a^*,b^*) + j_1(a^*,b^*) - 1] - k^2(j_2(a^*,b^*) - 2j_1(a^*,b^*)\lambda + \lambda^2 j_0(a^*,b^*)] \}$$

…(14) The Relative Efficiency of estimator $\tilde{\theta}_i$ (i=1,2) with respect to the classical estimator ($\hat{\theta}$) is defined as below

$$R.Eff(\tilde{\theta}_i \mid \theta, R) = \frac{MSE(\hat{\theta})}{MSE(\tilde{\theta}_i)} = \frac{\theta^2/n}{MSE(\tilde{\theta}_i)}$$

…(15)

4. Numerical Results

The computations of relative Efficiency [R.Eff(·)] and Bias ratio [B(·)] were used for the estimators $\tilde{\theta}_1$ and $\tilde{\theta}_2$. These computations were performed for α = 0.01,0.05,0.1, λ = 0.25(0.25)2, n = 4,6,8,12 and k = 0.1,0.3,0.5.
Some of these computations are given in four annexed tables. The observation mentioned in the tables lead to the following results:

1. **R.Eff(·) of $\tilde{\theta}_i$ (i=1,2) are adversely proportional with small value of α, k and n.**
2. **R.Eff(·) of $\tilde{\theta}_i$ (i=1,2) are maximum when $\theta = \theta_0$ ($\lambda = 1$) for all α, k and n.**
3. **R.Eff of θ_2 is better than R.Eff(·) of $\tilde{\theta}_1$ specialy when $\theta = \theta_0$.**
4. **Bias ratio [B(·)] of $\tilde{\theta}_i$ (i=1,2) $[B(\cdot) = \frac{Bias(\tilde{\theta}_i)}{\theta}, i = 1,2]$ are reasonably small when θ □ θ_0, otherwise start to be maximum for all α and n.**
5. **Bias ratio [B(·)] of $\tilde{\theta}_i$ (i=1,2) are at most increasing function with α.**
6. **Effictive Interval [the values of λ that makes R.Eff. greater than one] for $\tilde{\theta}_i$ (i = 1,2) is [0.5,1.5].**

7. $\tilde{\theta}_2$ **has higher R.Eff. with small value of k.**

   i.e. R.Eff. of θ_2 **is decreases function w.r.t. k.**

8. **Bias ratio of** $\tilde{\theta}_2$ **is increasing function with respect to k and sample size n.**

9. **The suggested estimators** $\tilde{\theta}_i$ **(i=1,2) are more efficient than the estimators introduced by [8], [9] and [10] in the sense of mean square error and Relative efficiency.**

5. Conclusions

1.The proposed estimator $\tilde{\theta}_i$ **(i=1,2) are consistent .**

i.e $\lim_{n\to\infty} MSE(\tilde{\theta}_i \mid \theta, R) = 0$

2. The proposed estimator $\tilde{\theta}_i$ **(i=1,2) are dominate to** $\hat{\theta}$**.**

i.e $\lim_{n\to\infty}[MSE(\tilde{\theta}_i \mid \theta, R) - MSE(\hat{\theta})] \le 0$

3. The considerd estimator $\tilde{\theta}_i$ **(i=1,2) has hiegh efficiency when** $\theta \approx \theta_0 (\lambda = 1)$**.**

4. The considerd estimator $\tilde{\theta}_i$ **(i=1,2) has lower bias when** $\theta \approx \theta_0 (\lambda = 1)$**.**

References:

1. **Epstein, B. and Sobcl, M., (1984), "Some Theorems to Life Testing from an Exponential Distribution", Annals of Mathematical Statistics, 25.**
2. **Thompson,J.R., (1968), "Some Shrinkage Techniques for Estimating the Mean", J. Amer. Statist. Assoc, 63, p.113-122.**
3. **Al-Jubori, A.N., (2000), "Preliminary Test Single Stage Shrrunken Estimator for The Parameters of Simple Linear Regerssion Model", Ibn Al-Haitham J.for Pure and Applied Sci., 13(3), pp. 65-73.**
4. **Al-Juboori,A.N., (2001), "On Shrinkage Techniques for Estimating the Variance of Normal Distribution Using Stein-Type Estimator", Ibn-Al-Haitham J. for Pure and Applied Sciences, 14 (4B), p.59-66.**
5. **Al-Juboori,A.N., (2002), "On Shrunken Estimators for the Parameters of Simple Linear Regression Model", Ibn-Al-Haitham J. for Pure and Applied Sciences, 15 (4A), p.60-67.**
6. **Kalaf,B.A., (2007),"An Efficient Shrinkage Estimators for the Mean of Normal Distribution", M.Sc. Thesis, Baghdad Univ. College of Education (Ibn-Al-Haitham), Baghdad, Iraq.**
7. **Al-Juboori,A.N. and Mohammad, M.A., (2008), "Preliminary Test Bayesian Shrunken Estimators for the mean of Normal Distribution with Known Variance"", Dyala, Jour., 31, pp.99-108.**
8. **Mehta,J.S. and Srinivasan,R., (1971), "Estimation of the Mean by Shrinkage to a Point", J. Amer. Statist. Assoc., 66, p.86-90.**
9. **Bhattachrya, S.K. and Srivastava, V. K., (1974), "A preliminary Test Procedure in Life Testing", J. Amer. Statist. Associ., 69, pp.726-729.**
10. **Pandey, B.N. and Srivastava, R., (1985), "On Shrinkage Estimation of Exponential Scale Parameter", IEEE Trans, Reli., R.34, pp. 224-226.**

Table (1) Shown Bias Ratio [B(·)] and Relative Effenciency [R.Eff.(·)] of $\tilde{\theta}_1$

α	λ / n		0.25	0.75	1	1.25	1.75	2
0.01	4	R.Eff(·) B(·)	0.41924 (0.44121)	3.3456 (0.2606)	9.3931 (0.035483)	2.5037 (0.16552)	0.4844 (0.46998)	0.31292 (0.56854)
	6	R.Eff(·) B(·)	0.33482 (0.38347)	2.2162 (0.26314)	5.1046 (0.054174)	1.6525 (0.1648)	0.44508 (0.25713)	0.34161 (0.26193)
	8	R.Eff(·) B(·)	0.2946 (0.33019)	1.6589 (0.26486)	3.5676 (0.070868)	1.379 (0.054027)	0.53456 (0.11125)	0.49468 (0.089647)
0.05	4	R.Eff(·) B(·)	0.48505 (0.2853)	2.396 (0.28444)	3.7688 (0.10626)	1.9607 (0.026597)	0.69944 (0.15839)	0.54404 (0.17415)
	6	R.Eff(·) B(·)	0.41925 (0.23496)	1.7427 (0.2858)	2.8176 (0.12389)	1.6932 (0.025157)	0.85668 (0.033818)	0.78732 (0.030896)
	8	R.Eff(·) B(·)	0.39365 (0.19026)	1.398 (0.28361)	2.5032 (0.12857)	1.6863 (0.044191)	1.0242 (0.000678)	0.97988 (0.002065)

Table (2) Shown Bias Ratio [B(·)] and Relative Effenciency [R.Eff.(·)] of $\tilde{\theta}_2$
when k = 0.1

α	λ / n		0.25	0.75	1	1.25	1.75	2
0.01	4	R.Eff(·) B(·)	0.2307 (0.86912)	1.9905 (0.30106)	939.31 (0.003548)	2.0564 (0.29155)	0.23005 (0.872)	0.13019 (1.1569)
	6	R.Eff(·) B(·)	0.15494 (0.86335)	1.317 (0.30131)	143.7 (0.005417)	1.3917 (0.28565)	0.15762 (0.85071)	0.089468 (1.1262)
	8	R.Eff(·) B(·)	0.11720 (0.85802)	0.98389 (0.30149)	78.107 (0.007087)	1.0619 (0.2804)	0.12057 (0.8361)	0.068309 (1.109)
0.05	4	R.Eff(·) B(·)	0.23545 (0.85353)	1.969 (0.30344)	376.88 (0.010626)	2.1627 (0.27766)	0.23967 (0.84084)	0.13543 (1.1174)
	6	R.Eff(·) B(·)	0.15803 (0.8484)	1.3044 (0.30358)	116.97 (0.012389)	1.4644 (0.27248)	0.16253 (0.82838)	0.09165 (1.1031)
	8	R.Eff(·) B(·)	0.11925 (0.84403)	0.97632 (0.30336)	71.455 (0.012858)	1.105 (0.27058)	0.12247 (0.82507)	0.06895 (1.1002)

Table (3) Shown Bias Ratio [B(·)] and Relative Effenciency [R.Eff.(·)] of $\tilde{\theta}_2$ when k = 0.3

α	n \ λ		0.25	0.75	1	1.25	1.75	2
0.01	4	R.Eff(·) B(·)	0.24254 (1.1074)	1.9652 (0.40318)	104.37 (0.010645)	2.0922 (0.37466)	0.2421 (1.116)	0.13884 (1.4706)
	6	R.Eff(·) B(·)	0.16482 (1.09)	1.2344 (0.40394)	15.967 (0.016252)	1.3787 (0.35695)	0.17355 (1.0521)	0.10092 (1.3786)
	8	R.Eff(·) B(·)	0.12606 (1.0741)	0.8995 (0.40446)	8.6785 (0.02126)	1.0575 (0.34121)	0.13805 (1.0084)	0.080044 (1.3269)
0.05	4	R.Eff(·) B(·)	0.25657 (1.0606)	1.8885 (0.41033)	41.876 (0.03188)	2.3279 (0.33298)	0.27168 (1.0225)	0.15581 (1.3522)
	6	R.Eff(·) B(·)	0.17444 (1.0455)	1.1969 (0.41075)	12.997 (0.037166)	1.5599 (0.31745)	0.19074 (0.98515)	0.10897 (1.3093)
	8	R.Eff(·) B(·)	0.13328 (0.13328)	0.87903 (0.41008)	7.9394 (0.03857)	1.1748 (0.31174)	0.14525 (0.9752)	0.082585 (1.3006)

Table (4) Shown Bias Ratio [B(·)] and Relative Effenciency [R.Eff.(·)] of $\tilde{\theta}_2$ when k = 0.5

α	n \ λ		0.25	0.75	1	1.25	1.75	2
0.01	4	R.Eff(·) B(·)	0.24566 (1.3456)	1.9323 (0.5053)	37.572 (0.017742)	2.0197 (0.45776)	0.24792 (1.36)	0.14418 (1.7843)
	6	R.Eff(·) B(·)	0.16798 (1.3167)	1.1063 (0.50657)	5.7479 (0.027087)	1.2123 (0.42824)	0.18235 (1.2536)	0.10938 (1.631)
	8	R.Eff(·) B(·)	0.12963 (1.2901)	0.77486 (0.50743)	3.1243 (0.035434)	0.90064 (0.40201)	0.14978 (1.1806)	0.089977 (1.5448)
0.05	4	R.Eff(·) B(·)	0.26625 (1.2677)	1.7887 (0.51722)	15.075 (0.053129)	2.206 (0.3883)	0.29347 (1.2042)	0.17218 (1.587)
	6	R.Eff(·) B(·)	0.18286 (1.2425)	1.0509 (0.51791)	4.6789 (0.061944)	1.384 (0.36242)	0.21224 (1.1419)	0.12445 (1.5154)
	8	R.Eff(·) B(·)	0.14122 (0.14122)	0.7477 (0.7477)	2.8582 (2.8582)	1.0214 (1.0214)	0.16337 (0.16337)	0.095117 (0.095117)