Comparing Different Estimators for the shape Parameter and the Reliability function of Kumaraswamy Distribution

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Abstract

In this paper, we used maximum likelihood method and the Bayesian method to estimate the shape parameter (θ), and reliability function (R(t)) of the Kumaraswamy distribution with two parameters λ , θ (under assuming the exponential distribution, Chi-squared distribution and Erlang-2 type distribution as prior distributions), in addition to that we used method of moments for estimating the parameters of the prior distributions. Bayes estimators derived under the squared error loss function. We conduct simulation study, to compare the performance for each estimator, several values of the shape parameter (θ) from Kumaraswamy distribution for data-generating, for different samples sizes (small, medium, and large). Simulation results have shown that the Best method is the Bayes estimation according to the smallest values of mean square errors(MSE) for all samples sizes (n).

Keywords: **Kumaraswamy distribution, the shape parameter, reliability function, maximum likelihood, Bayes method, prior distributions (exponential distribution, chi-squared distribution and Erlang-2 type distribution), the squared error loss function.**

1.Introduction

In this paper, we use the new probability distribution to handle the problem of survival data. Motivated by research developed in recent years. I introduce the Kumaraswamy distribution which is a member of continuous probability distributions, which was constructed by Kumaraswamy [5] (1980). This distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures, hydrological data, etc. We mention some of the studies in a brief manner:

Gholizadeh and Gholizadeh ,Khalilpor and Hadian [2] (2011) used maximum likelihood and Bayes estimation of the shape parameter θ, reliability R(t) and failure rate H(t) functions of the Kumaraswamy distribution based on a progressively type-II censored sample. They derived Bayesian estimators under the squared error loss, Precautionary and LINEX loss functions ,using simulation technique, to assess the statistical performances of these estimates.

Gholizadeh and Shirazi and Mosalmanzadeh [3] (2011) obtained classical and Bayesian estimators for the shape parameter θ of the Kumaraswamy distribution using un- groped data. They derived the Bayes estimators under the squared error loss, Precautionary and General entropy loss function. They considered both point and interval estimators using simulation technique.

In (2014) Mohie Eldin and Khalil and Amein, [6] estimated the parameters of the Kumaraswamy distribution based on general progressive type-II censoring data by using the Maximum Likelihood and Bayesian Methods. In the Bayesian Method, the two parameters are assumed to be random variables and estimators for the parameters are obtained using the well-known Squared Error Loss (SEL) function. The simulating study was conducted in order to compare the performance of the Bayes estimators with MLE for different sample sizes and censoring schemes.

Sultan and Ahmad [9] (2015) estimated the shape parameter of Kumaraswamy Distribution using various Bayesian approximation techniques like Normal approximation, Lindley's Approximation, Tierney and Kadane (T-K) Approximation. They obtained the Baye's estimates under different informative and non-informative priors of the shape parameter of Kumaraswamy Distribution.

Simbolon and Fithriani and Nurrohmah [8] (2016) estimated the shape parameter of Kumaraswamy Distribution using the Maximum Likelihood (ML) and Bayes methods. They derived the Bayes estimators under two loss functions, the Square Error Loss (SEL) and the Precautionary Loss Function (PLF). They compared the shape parameter estimation by Maximum Likelihood (ML) method and Bayes method based on the Mean Square Error (MSE). The best estimator is the smaller MSE value.

Ghosh [4] (2017) derived the maximum likelihood estimate of the parameters of the Kumaraswamy distribution by using Newton-Raphson as well as The Expectation-Maximization(EM) algorithm method. Also, he provided the Tierney and Kadane's approximation, to compute the Bayes estimates of the unknown parameters. He compared the estimation procedures via Markov Chain Monte Carlo simulations in terms of their average biases and Mean Squared Errors.

Salman [7] (2017) used different estimation methods, to estimate the parameters of the Kumaraswamy distribution by using Maximum Likelihood, Moment estimator, and the Maximum Likelihood of ordered observation. She compared the estimation procedures through simulation using different sample size and a different set of initial values of the parameters, then comparing the results using statistical measure mean square error (MSE).

So in this paper, we try to find the best estimation for the shape parameter(θ) and for Reliability function ($R(t)$) of the Kumaraswamy distribution which means the probability of surviving at least till age t. According to the smallest value of Mean Square Errors (MSE) was calculated to compare maximum likelihood method and we derive Bayes estimators after derived the posterior distributions under three types of prior distributions to get Bayes estimation :The exponential distribution, Chi-squared distribution, Erlang-2 distribution when the Bayesian estimation based on the squared error loss function .Several cases from Kumaraswamy distribution for data-generating ,for different sample sizes (small, medium, and large).The results were obtained by using simulation technique, Programs written using MATLAB-R2015b program were used.

2. The Kumaraswamy Distribution

Let us consider $t_1, t_2, \ldots, t_n$ is a random sample of n independent observations from a Kumaraswamy distribution having the probability density function (pdf) define as [5]:

$$f_T(t)= \theta\,\lambda\, t^{\lambda-1}(1-t^{\lambda})^{\theta-1} \quad , \; 0 < t < 1, \lambda , \theta > 0 \qquad \ldots (1)$$

Where λ , θ is the shape parameters, respectively. Here we assume that λ the parameter is known. So, the cumulative distribution function (cdf) is given as;

$$F_T(t)=1-(1-t^{\lambda})^{\theta} \quad , \; 0 < t < 1, \lambda , \theta > 0 \qquad \ldots (2)$$

And the reliability function is

$$R_T(t)=(1-t^{\lambda})^{\theta} \quad , \; 0 < t < 1, \lambda , \theta > 0 \qquad \ldots (3)$$

3. Maximum Likelihood Estimation Method

Let (t_1, t_2 ,…, t_n) be a random sample of size n with probability density function given in equation (1) and likelihood function from the Kumaraswamy distribution given in (1) will be as follows [1]:

$$L(\theta) = \prod_{i=1}^{n} f(t_i) = \theta^n \lambda^n \prod_{i=1}^{n} t^{\lambda-1} \prod_{i=1}^{n} (1-t^{\lambda})^{\theta-1} \quad \dots(4)$$

The log-likelihood function for equation (4) is

$$\log L(\theta) = n \log(\theta) + n \log(\lambda) + (\lambda-1) \sum_{i=1}^{n} \log(t_i) + (\theta-1) \sum_{i=1}^{n} \log(1-t_i^{\lambda}) \quad \dots(5)$$

then to obtain the maximum likelihood estimators for θ,we derive equation (5) with respect to θ as follow

$$(\partial/\partial\theta) \log L(\theta) = (n/\theta) + \sum_{i=1}^{n} \log(1-t_i^{\lambda}) \quad \dots(6)$$

By setting $(\partial/\partial\theta) \log L(\theta) = 0$, then the $\hat{\theta}_{MLE}$ is

$$\hat{\theta}_{ML} = \left(-n / \sum_{i=1}^{n} \log(1-t_i^{\lambda})\right) \quad \dots(7)$$

4. Bayes Estimation Method

In this section, we use Bayes estimation to estimate the unknown parameter θ. Let $(t_1, t_2, \dots t_n)$ be a random sample of size n with probability density function given in equation (1) and likelihood function from the Kumaraswamy pdf given in equation (4).Here we assumed the unknown parameter θ is a random variable according to the following three types of informative priors: Exponential distribution[12], Chi-squared distribution[10] and Erlang-2 type distribution[11], as priors distribution. So in this paper, we derive the posterior distributions for the unknown parameter θ using the above prior distributions to get Bayes estimation.

4.1 The Posterior Distribution Using Different Priors

In this section , we discuss the posterior distributions .we assumed that θ follows three types of prior distributions with pdf as given in table -1:

Table -1: The Three Types of Prior's Distribution ($P_i(\theta)$) With pdf for θ.

prior distribution	$P_i(\theta)$, i=1,2,3
θ~ Exponential distribution (a) where a is the rate	$P_1(\theta) = a \exp(-a\theta)$, for a,θ > 0
θ~chi-square(v)	$P_2(\theta) = \frac{1}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{(1/2)} \theta^{(v/2)-1} \exp(-\frac{1}{2}\theta)$, for v,θ > 0
θ~Erlang -2 type(λ_1)	$P_3(\theta) = \lambda_1^2 \theta \exp(-\lambda_1 \theta)$, for λ_1,θ > 0

where Γ(.) denotes the gamma function, which has closed-form values for integer v.

We used the method of moments [1] to estimate the parameters for prior distributions with pdf as given in table -1(for more details see Appendix _C).

Table -2: The Estimate the Parameters for Prior Distributions

prior distribution	The estimate the parameters for prior distributions
θ~ Exponential distribution (a) where a is the rate	$\hat{a} = n / \sum_{i=1}^{n} \theta_i$
θ~chi-square(v)	$\hat{v} = \sum_{i=1}^{n} \theta_i / n = \bar{\theta}$
θ~Erlang -2 type(λ_1)	$\hat{\lambda}_1 = 2/\bar{\theta} = 2n / \sum_{i=1}^{n} \theta_i$

Then the posterior distribution of θ for the given the data $(t_1, t_2, \ldots t_n)$ is given by[1]:

$$P(\theta \backslash t)= \frac{L(\theta)P(\theta)}{\int_{\theta} L(\theta)P(\theta)\,d\theta} \qquad \ldots(8)$$

To derive the posterior distribution , we use the equation (4) and the prior distribution with pdf as given in table -1(for each $P_i(\theta)$ for i=1,2,3) in the equation (8), we get the posterior distributions for the unknown parameter θ are derived using the following three types of prior distributions (for more details see Appendix _A).

Table -3: the Posterior Distributions ($P(\theta\backslash t)$) for the Unknown Parameter θ are derived Using the following Three types of Prior Distributions

prior distribution	The posterior distribution($P(\theta\backslash t)$)
θ~ Exponential distribution (a) where a is the rate	$P_1(\theta\backslash t)= \frac{1}{\Gamma(n+1)} \left(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+1} \theta^{(n+1)-1} \exp\left(-\theta\left(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)\right)$, for $a,\theta,\lambda,n > 0$
θ~chi-square(v)	$P_2(\theta\backslash t) = \frac{1}{\Gamma(n+(0.5v))} \left(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+(0.5v)} \theta^{(n+(0.5v))-1} \exp\left(-\theta\left(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)\right)$, for $v,\theta,\lambda,n > 0$
θ~Erlang -2 type (λ_1)	$P_3(\theta\backslash t)= \frac{1}{\Gamma(n+2)} \left(\lambda_1-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+2} \theta^{n+2-1} \exp\left(-\theta\left(\lambda_1-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)\right)$ for $\lambda_1,\theta,\lambda,n > 0$

4.2 Bayes 'Estimators

First: Bayes 'Estimators for the shape parameter (θ)

The objective of this section is to find Bayesian estimators of the shape parameter (θ) of Kumaraswamy distribution under the squared error loss function.

$$L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2 \qquad \ldots(9)$$

Where $\hat{\theta}$ is an estimator for θ was considered with three types of prior distributions. After simplified steps , we get Baye estimators of θ denoted by $\hat{\theta}_{SE}$ for the above prior as follows:-

$$\hat{\theta}_{SE}=E(\theta\backslash t)=\int_{\theta} \theta\, P(\theta\backslash t)\, d\theta \qquad \ldots(10)$$

So, the following results are the derivations of these estimators under the squared error loss function with three types of prior distributions (for more details see Appendix _B).

Table -4: The Estimators ($\hat{\theta}_{SE}$)Under the Squared Error Loss Function Using Three Types of Prior Distributions

prior to distribution	$\hat{\theta}_{SE}=E(\theta\backslash t)=\int_{\theta} \theta\, P(\theta\backslash t)\, d\theta$
θ~ Exponential distribution (a) where a is the rate	$\hat{\theta}_{SE1}=(n+1)/(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))$, for a,θ, λ, n > 0
θ~chi-square(v)	$\hat{\theta}_{SE2}=(n+0.5v)/(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))$, for v,θ , λ, n > 0
θ~Erlang -2(λ_1)	$\hat{\theta}_{SE3}=(n+2)/(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))$, for λ_1,θ, λ, n > 0

Second: Bayes 'Estimators for the Reliability function (R(t))

Bayesian estimators of the reliability function (R=R(t)) of Kumaraswamy distribution under the squared error loss function.

$$L(\hat{R},R)=(\hat{R}-R)^2 \qquad \ldots(11)$$

$$Risk=\int_{\theta}(\hat{R}-R)^2\, P(\theta\backslash t)\, d\theta$$

Where $\hat{R}(t)$ is an estimator for R(t) was considered with three types of prior distributions. After simplified steps , we get Baye estimators of R(t) denoted by $\hat{R}_{SE}(t)$ for the above prior as follows

$$\hat{R}_{SE}(t)=E(R \backslash t)= \int_{\theta} R(t)P(\theta \backslash t)\, d\theta \qquad \ldots(12)$$

So, the following results are the derivations of these estimators under the squared error loss function with three types of prior distributions (for more details see Appendix _B).

Table -4: the Estimators ($\hat{R}_{SE}$)Under the Squared Error Loss Function Using Three Types of Prior Distributions

prior distribution	$\hat{R}_{SE}=E(R \backslash t)= \int_{\theta} R\, P(\theta \backslash t)\, d\theta$
θ~ Exponential distribution (a) where a is the rate	$\hat{R}_{SE1}= [(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda}))/(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda})-\log(1-t^{\lambda}))]^{(n+1)}$ for a,θ, λ, n > 0
θ~chi-square(v)	$\hat{R}_{SE2}=[(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}))/(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})-\log(1-t^{\lambda}))]^{(n+0.5v)}$ for v,θ , λ, n > 0
θ~Erlang -2(λ_1)	$\hat{R}_{SE3}= [(\lambda_1-\sum_{i=1}^{n} \log(1-t_i^{\lambda}))/ (\lambda_1-\sum_{i=1}^{n} \log(1-t_i^{\lambda})-\log(1-t^{\lambda}))]^{(n+2)}$ for λ_1,θ, λ, n > 0

4. Simulation Study

In simulation study, we chose Four sample size(25,50,100,150) to represent small, moderate and large sample size, we generated data from a Kumaraswamy distribution for the shape parameter, according to the following cdf $F_T(t)=1-(1-t^{\lambda})^{\theta}$, by setting $F_i = u_i$ where $u_i \sim$ uniform dist.$^n(0,1)$, we have $u_i=1-(1-t^{\lambda})^{\theta}$ then $t_i=(1-((1-u_i)^{(1/\theta)})^{(1/\lambda)})$.We consider randomly several values for the shape parameter θ= 0.5,1,1.5 and λ = 0.5,1,1.5 of Kumaraswamy distribution .

Also , we chose randomly the values for the distributions (exponential (a=3,4,5) , Chi-square (v=3,5,7)and Erlang-2 type (λ_1 =3,4,5)) as prior distribution for θ. The number of replication use is (L=1000) for each sample size (n). The simulation program is written using matlab-R2015b program. After estimating the parameters for θ and R(t) under the squared error loss function with three different priors, the mean square error (MSE) was calculated to compare the methods of estimation. Using the following:-

$$MSE(\hat{\theta})= \frac{1}{L}\sum_{L=1}^{1000} (\hat{\theta}_L - \theta)^2 \qquad \ldots(13)$$

$$MSE(\hat{R}(t))= \frac{1}{L}\sum_{L=1}^{1000} (\hat{R}_L(t)- R(t))^2 \qquad \ldots(14)$$

The results of the simulation study are summarized and tabulated in tables (4.1) - (4.6).Using ML and Bayes estimation method under the squared error loss function with three different priors. According to criterion is the best method that gives the smallest value of (MSE).

Table 4.1: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

parameters			Estimate for θ ($\hat{\theta}$)				MSE($\hat{\theta}$)			
			Sample Size(n)				Sample Size(n)			
			25	50	100	150	25	50	100	150
θ	λ		MLE							
0.5	0.5		0.52328	0.51002	0.50309	0.50366	0.01237	0.00524	0.00241	0.00167
	1.0		0.51937	0.51168	0.50431	0.50151	0.01241	0.00561	0.00265	0.00150
	1.5		0.51937	0.50733	0.50582	0.50351	0.01292	0.00498	0.00252	0.00183
θ	λ	a	Exponential distribution ($P_1(\theta \mid t)$)							
0.5	0.5	3	0.51084	0.50449	0.5005	0.50193	0.00995	0.00473	0.00230	0.00161
		4	0.50064	0.49946	0.498	0.50025	0.00904	0.00452	0.00226	0.00159
		5	0.49086	0.49453	0.49554	0.49859	0.00841	0.00437	0.00223	0.00157
0.5	1.0	3	0.50722	0.50607	0.50168	0.49981	0.01001	0.00506	0.00253	0.00146
		4	0.49716	0.501	0.49918	0.49815	0.00915	0.00482	0.00247	0.00145
		5	0.4875	0.49603	0.4967	0.4965	0.00857	0.00464	0.00244	0.00144
0.5	1.5	3	0.50717	0.50191	0.50317	0.50178	0.01044	0.00453	0.00240	0.00177
		4	0.49709	0.49694	0.50065	0.5001	0.00956	0.00436	0.00234	0.00175
		5	0.48742	0.49206	0.49816	0.49844	0.00896	0.00424	0.00229	0.00172
θ	λ	v	Chi-squared distribution ($P_2(\theta \mid t)$)							
0.5	0.5	3	0.54869	0.52261	0.50934	0.50784	0.01507	0.00584	0.00253	0.00174
		5	0.5694	0.53275	0.51436	0.51119	0.0185	0.00662	0.00270	0.00182
		7	0.5901	0.5429	0.51938	0.51454	0.02281	0.00760	0.00292	0.00193
0.5	1.0	3	0.51454	0.5243	0.51057	0.50567	0.00193	0.00627	0.00280	0.00155
		5	0.56518	0.53448	0.5156	0.50901	0.01815	0.00709	0.00298	0.00162
		7	0.58573	0.54466	0.52063	0.51235	0.02229	0.00813	0.00322	0.00172
0.5	1.5	3	0.54462	0.51986	0.5121	0.50769	0.01545	0.00551	0.00268	0.00190
		5	0.56517	0.52996	0.51714	0.51104	0.01874	0.00622	0.00288	0.00199
		7	0.58572	0.54005	0.52219	0.51439	0.02292	0.00713	0.00313	0.00210
θ	λ	λ_1	Erlang-2 type distribution ($P_3(\theta \mid t)$)							
0.5	0.5	2	0.54154	0.51962	0.50799	0.50695	0.01327	0.00549	0.00246	0.00170
		4	0.5199	0.50925	0.50294	0.50357	0.01014	0.00479	0.00231	0.00162
		6	0.49998	0.4993	0.49798	0.50023	0.00829	0.00434	0.00222	0.00157
0.5	1.0	2	0.53763	0.52126	0.5092	0.5048	0.01313	0.00589	0.00271	0.00152
		4	0.51628	0.51082	0.50412	0.50145	0.01013	0.00513	0.00254	0.00146
		6	0.49661	0.5008	0.49914	0.49814	0.00838	0.00462	0.00243	0.00143
0.5	1.5	2	0.5376	0.51694	0.51072	0.5068	0.01363	0.00519	0.00260	0.00186
		4	0.51621	0.50668	0.50561	0.50342	0.01056	0.00456	0.00242	0.00178
		6	0.49652	0.49683	0.5006	0.50008	0.00877	0.00418	0.00229	0.00172

Table 4.2: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

parameters			Estimate for θ ($\hat{\theta}$)				MSE($\hat{\theta}$)			
			Sample Size(n)				Sample Size(n)			
			25	50	100	150	25	50	100	150
θ	λ		MLE							
1	0.5		1.034	1.0167	1.0046	1.007	0.04995	0.01980	0.00974	0.00733
	1.0		1.0397	1.0132	1.0098	1.0049	0.04997	0.02260	0.01007	0.00591
	1.5		1.0425	1.018	1.0113	1.008	0.04588	0.02046	0.01046	0.00685
θ	λ	a	Exponential distribution ($P_1(\theta \mid t)$)							
1	0.5	3	0.95246	0.97638	0.98468	0.99358	0.03387	0.01647	0.00902	0.00685
		4	0.91772	0.95775	0.97509	0.98704	0.03385	0.01650	0.00907	0.00680
		5	0.8855	0.93982	0.96568	0.98059	0.03646	0.01726	0.00930	0.00684
1	1.0	3	0.95717	0.97311	0.98959	0.99159	0.03328	0.01894	0.00912	0.00557
		4	0.9221	0.95455	0.9799	0.98509	0.03301	0.01890	0.00906	0.00558
		5	0.88959	0.9367	0.9704	0.97867	0.03543	0.01959	0.00920	0.00568
1	1.5	3	0.95983	0.97758	0.99101	0.99453	0.03046	0.01692	0.00941	0.00637
		4	0.92467	0.95889	0.98129	0.98798	0.03044	0.01687	0.00932	0.00632
		5	0.89206	0.94091	0.97176	0.98152	0.03304	0.01756	0.00942	0.00636
θ	λ	v	Chi-squared distribution ($P_2(\theta \mid t)$)							
1	0.5	3	1.0729	1.0364	1.0145	1.0137	0.05536	0.02119	0.01003	0.00752
		5	1.1133	1.0566	1.0245	1.0204	0.06675	0.02384	0.01061	0.00784
		7	1.1538	1.0767	1.0345	1.027	0.08155	0.02732	0.01140	0.00826
1	1.0	3	1.0786	1.0329	1.0197	1.0116	0.05586	0.02388	0.01046	0.00606
		5	1.1194	1.053	1.0297	1.0183	0.06774	0.02650	0.01115	0.00634
		7	1.1601	1.073	1.0398	1.0249	0.08308	0.02994	0.01205	0.00670
1	1.5	3	1.0816	1.0378	1.0212	1.0146	0.05197	0.02192	0.01088	0.00704
		5	1.1224	1.0579	1.0313	1.0213	0.06378	0.02465	0.01161	0.00738
		7	1.1632	1.0781	1.0413	1.028	0.07905	0.02821	0.01255	0.00780
θ	λ	λ_1	Erlang-2 type distribution ($P_3(\theta \mid t)$)							
1	0.5	2	1.0281	1.0153	1.0043	1.0068	0.04089	0.01815	0.00935	0.00713
		4	0.95302	0.97653	0.98474	0.99358	0.03141	0.01585	0.00885	0.00676
		6	0.88843	0.94065	0.96593	0.98067	0.03429	0.01668	0.00913	0.00675
1	1.0	2	1.0334	1.0119	1.0094	1.0048	0.04097	0.02067	0.00966	0.00575
		4	0.95757	0.97327	0.9896	0.99161	0.03086	0.01821	0.00894	0.00550
		6	0.89241	0.93755	0.9706	0.97877	0.03332	0.01891	0.00903	0.00561
1	1.5	2	1.0363	1.0166	1.0108	1.0078	0.03781	0.01875	0.01003	0.00666
		4	0.96023	0.97769	0.99101	0.99453	0.02829	0.01628	0.00923	0.00629
		6	0.89487	0.94171	0.97195	0.9816	0.03109	0.01697	0.00925	0.00628

Table 4.3: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

parameters			Estimate for θ ($\hat{\theta}$)				MSE($\hat{\theta}$)			
			Sample Size(n)				Sample Size(n)			
			25	50	100	150	25	50	100	150
θ	λ		MLE							
1.5	0.5		1.5595	1.5351	1.5074	1.5077	0.11245	0.05006	0.02268	0.01396
	1.0		1.5637	1.522	1.5202	1.5062	0.10323	0.04487	0.02184	0.01516
	1.5		1.5519	1.5291	1.5158	1.508	0.10882	0.04629	0.02579	0.01564
θ	λ	a	Exponential distribution ($P_1(\theta \mid t)$)							
1.5	0.5	3	1.3583	1.4315	1.456	1.4731	0.07634	0.03995	0.02121	0.01322
		4	1.289	1.3918	1.4352	1.4587	0.08987	0.04317	0.02239	0.01371
		5	1.2266	1.3542	1.4149	1.4447	0.11176	0.04940	0.02441	0.01461
1.5	1.0	3	1.3621	1.4204	1.4678	1.4716	0.07077	0.03849	0.0193	0.01440
		4	1.2926	1.3814	1.4466	1.4573	0.08483	0.04279	0.02007	0.01489
		5	1.23	1.3444	1.4261	1.4433	0.10709	0.04993	0.02174	0.01579
1.5	1.5	3	1.3529	1.4265	1.4637	1.4733	0.07596	0.03825	0.02304	0.01471
		4	1.2842	1.3871	1.4426	1.459	0.09029	0.04206	0.02378	0.01514
		5	1.2223	1.3498	1.4221	1.4449	0.11275	0.04880	0.02540	0.01598
θ	λ	v	Chi-squared distribution ($P_2(\theta \mid t)$)							
1.5	0.5	3	1.601	1.5568	1.5185	1.5151	0.11714	0.05184	0.02295	0.01412
		5	1.6614	1.587	1.5334	1.5251	0.14122	0.05810	0.02417	0.01471
		7	1.7218	1.6173	1.5484	1.5351	0.1729	0.06622	0.02585	0.01550
1.5	1.0	3	1.6054	1.5437	1.5312	1.5136	0.10871	0.04614	0.02239	0.01531
		5	1.666	1.5737	1.5463	1.5236	0.13266	0.05139	0.02399	0.01588
		7	1.7265	1.6037	1.5614	1.5336	0.16422	0.05847	0.02604	0.01665
1.5	1.5	3	1.5935	1.5508	1.5269	1.5154	0.11272	0.04784	0.02624	0.01581
		5	1.6536	1.5809	1.5419	1.5254	0.13557	0.05358	0.02778	0.01642
		7	1.7137	1.611	1.557	1.5354	0.16595	0.06117	0.02978	0.01724
θ	λ	λ_1	Erlang-2 type distribution ($P_3(\theta \mid t)$)							
1.5	0.5	2	1.491	1.5025	1.4922	1.4975	0.07639	0.04124	0.02091	0.01318
		4	1.3386	1.4191	1.4494	1.4684	0.07497	0.03925	0.02111	0.01317
		6	1.2151	1.3446	1.409	1.4404	0.11409	0.05045	0.02483	0.01481
1.5	1.0	2	1.4951	1.4905	1.5044	1.496	0.07003	0.03767	0.01978	0.01435
		4	1.3423	1.4085	1.461	1.467	0.06995	0.03824	0.01909	0.01434
		6	1.2184	1.3351	1.42	1.439	0.10974	0.05123	0.02208	0.01598
1.5	1.5	2	1.4847	1.497	1.5002	1.4978	0.07408	0.03842	0.02352	0.01476
		4	1.3336	1.4143	1.4569	1.4686	0.07485	0.03782	0.02275	0.01462
		6	1.2111	1.3403	1.4161	1.4406	0.11514	0.05000	0.02567	0.01615

4.4: MSE of estimated Kumaraswamy reliability function under the squared error loss function.

parameters				R(t)	$(\hat{R}(t))$				$MSE(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					25	50	100	150	25	50	100	150
θ	λ	t			MLE							
0.5	0.5	0.5		0.5412	0.53048	0.53662	0.54012	0.53944	0.00464	0.00215	0.00104	0.00071
	1.0	0.5		0.70711	0.69964	0.70232	0.70544	0.70662	0.00267	0.00128	0.00062	0.00035
	1.5	0.5		0.80402	0.79819	0.80183	0.80217	0.80293	0.00148	0.00059	0.00030	0.00022
θ	λ	t	a		Exponential distribution ($P_1(\theta \mid t)$)							
0.5	0.5	0.5	3	0.5412	0.54187	0.54212	0.54281	0.54125	0.00381	0.00195	0.00100	0.00069
			4		0.54824	0.54536	0.54445	0.54235	0.00366	0.00192	0.00100	0.00069
			5		0.55444	0.54856	0.54607	0.54344	0.00360	0.00190	0.00100	0.00068
0.5	1.0	0.5	3	0.70711	0.7069	0.70584	0.70713	0.70773	0.00217	0.00116	0.00059	0.00034
			4		0.71165	0.70827	0.70835	0.70854	0.00206	0.00112	0.00058	0.00034
			5		0.71625	0.71066	0.70956	0.70934	0.00199	0.00109	0.00058	0.00034
0.5	1.5	0.5	3	0.80402	0.80306	0.80407	0.80328	0.80366	0.00120	0.00054	0.00029	0.00021
			4		0.8065	0.8058	0.80416	0.80424	0.00112	0.00052	0.00028	0.00021
			5		0.80983	0.8075	0.80503	0.80482	0.00107	0.00051	0.00028	0.00021
θ	λ	t	v		Chi-squared distribution ($P_2(\theta \mid t)$)							
0.5	0.5	0.5	3	0.5412	0.51879	0.53055	0.53704	0.53738	0.00492	0.00222	0.00105	0.00072
			5		0.50627	0.5241	0.53377	0.53518	0.00574	0.00243	0.00110	0.00075
			7		0.49406	0.51773	0.53052	0.533	0.00683	0.00271	0.00117	0.00078
0.5	1.0	0.5	3	0.70711	0.68949	0.69713	0.70283	0.70488	0.00295	0.00137	0.00064	0.00036
			5		0.67996	0.69229	0.7004	0.70325	0.00349	0.00152	0.0006	0.00037
			7		0.67056	0.68747	0.69797	0.70163	0.00421	0.00172	0.00072	0.00039
0.5	1.5	0.5	3	0.80402	0.79037	0.79787	0.80018	0.8016	0.00168	0.00064	0.00032	0.00022
			5		0.78342	0.79439	0.79842	0.80043	0.00200	0.00071	0.00034	0.00023
			7		0.77654	0.79092	0.79667	0.79926	0.00242	0.00081	0.00036	0.00025
θ	λ	t	λ_1		Erlang-2 type distribution ($P_3(\theta \mid t)$)							
0.5	0.5	0.5	2	0.5412	0.52276	0.53237	0.5379	0.53795	0.00446	0.00211	0.00103	0.00071
			4		0.53584	0.53895	0.54119	0.54016	0.00374	0.00193	0.00099	0.00069
			6		0.54825	0.54536	0.54444	0.54235	0.00340	0.00184	0.00098	0.00068
0.5	1.0	0.5	2	0.70711	0.69257	0.69854	0.70348	0.7053	0.00265	0.00130	0.00062	0.00035
			4		0.70246	0.70351	0.70594	0.70693	0.00216	0.00116	0.00059	0.00034
			6		0.71173	0.70832	0.70836	0.70854	0.00190	0.00107	0.00057	0.00034
0.5	1.5	0.5	2	0.80402	0.79266	0.79886	0.80065	0.80191	0.00150	0.00060	0.00031	0.00022
			4		0.79989	0.8024	0.80243	0.80309	0.00120	0.00054	0.00029	0.00021
			6		0.80661	0.80582	0.80417	0.80425	0.00103	0.00050	0.00027	0.00021

4.5: MSE of estimated Kumaraswamy reliability function under the squared error loss function.

parameters				R(t)	$\hat{R}(t)$				MSE($\hat{R}(t)$)			
					Sample Size(n)				Sample Size(n)			
					25	50	100	150	25	50	100	150
θ	λ	t			MLE							
1	0.5	0.5		0.29289	0.29054	0.2911	0.29336	0.29197	0.00504	0.00234	0.00121	0.00091
	1.0	0.5		0.5	0.49186	0.49805	0.49781	0.49899	0.00503	0.00250	0.00116	0.00069
	1.5	0.5		0.64645	0.63718	0.6426	0.64391	0.64461	0.00327	0.00155	0.00080	0.00053
θ	λ	t	a		Exponential distribution ($P_1(\theta \mid t)$)							
1	0.5	0.5	3	0.29289	0.32542	0.30924	0.30255	0.29816	0.00508	0.00234	0.00123	0.00090
			4		0.33802	0.31597	0.30602	0.30051	0.00583	0.00255	0.00129	0.00092
			5		0.35025	0.3226	0.30946	0.30284	0.00686	0.00284	0.00138	0.00096
1	1.0	0.5	3	0.5	0.52314	0.51385	0.50587	0.50437	0.00415	0.00232	0.00110	0.00067
			4		0.53516	0.52025	0.50922	0.50662	0.00451	0.00243	0.00112	0.00069
			5		0.54661	0.52649	0.51252	0.50885	0.00515	0.00262	0.00117	0.00071
1	1.5	0.5	3	0.64645	0.66187	0.65498	0.65016	0.64878	0.00246	0.00136	0.00074	0.00050
			4		0.67169	0.66022	0.65289	0.65063	0.00262	0.00140	0.00075	0.00050
			5		0.68095	0.66531	0.65558	0.65245	0.00296	0.00150	0.00077	0.00051
θ	λ	t	v		Chi-squared distribution ($P_2(\theta \mid t)$)							
1	0.5	0.5	3	0.29289	0.28562	0.2885	0.292	0.29107	0.00464	0.00225	0.00118	0.00090
			5		0.27274	0.2817	0.2885	0.28871	0.00489	0.00233	0.00119	0.00091
			7		0.26046	0.27506	0.28504	0.28638	0.00543	0.00250	0.00123	0.00093
1	1.0	0.5	3	0.5	0.48378	0.49375	0.49561	0.49751	0.00494	0.00247	0.00116	0.00069
			5		0.47089	0.48708	0.49221	0.49523	0.00560	0.00262	0.00120	0.00071
			7		0.45836	0.4805	0.48883	0.49295	0.00656	0.00286	0.00128	0.00074
1	1.5	0.5	3	0.64645	0.62908	0.63837	0.64175	0.64316	0.00338	0.00158	0.00082	0.00054
			5		0.61827	0.63285	0.63896	0.64129	0.00399	0.00173	0.00086	0.00055
			7		0.60766	0.62739	0.63618	0.63943	0.00482	0.00194	0.00092	0.00058
θ	λ	t	λ_1		Erlang-2 type distribution ($P_3(\theta \mid t)$)							
1	0.5	0.5	2	0.29289	0.29905	0.29547	0.29552	0.29344	0.00424	0.00213	0.00116	0.00088
			4		0.32443	0.30897	0.30247	0.29814	0.00475	0.00226	0.00121	0.00089
			6		0.34843	0.32209	0.30931	0.30278	0.00643	0.00274	0.00135	0.00094
1	1.0	0.5	2	0.5	0.49764	0.50062	0.49908	0.49982	0.00408	0.00226	0.00110	0.00067
			4		0.52256	0.51367	0.50583	0.50435	0.00387	0.00223	0.00108	0.00067
			6		0.54524	0.52608	0.51242	0.5088	0.00484	0.00253	0.00114	0.00070
1	1.5	0.5	2	0.64645	0.64088	0.64412	0.64462	0.64506	0.00264	0.00140	0.00076	0.00051
			4		0.66154	0.65489	0.65015	0.64878	0.00229	0.00131	0.00073	0.00049
			6		0.67997	0.66502	0.65551	0.65242	0.00278	0.00145	0.00076	0.00050

4.6: MSE of estimated Kumaraswamy reliability function under the squared error loss function.

parameters				R(t)	$(\hat{R}(t))$				$MSE(\hat{R}(t))$			
					Sample Size(n)				Sample Size(n)			
					25	50	100	150	25	50	100	150
θ	λ	t			MLE							
1.5	0.5	0.5		0.15851	0.15851	0.15723	0.15971	0.15865	0.00318	0.00163	0.00082	0.00050
	1.0	0.5		0.35355	0.34599	0.35184	0.35043	0.35331	0.00501	0.00246	0.00125	0.00088
	1.5	0.5		0.51976	0.513	0.51539	0.51742	0.51872	0.00466	0.00220	0.00127	0.00078
θ	λ	t	a		Exponential distribution ($P_1(\theta \mid t)$)							
1.5	0.5	0.5	3	0.15851	0.20596	0.182	0.17232	0.16713	0.00501	0.00207	0.00098	0.00057
			4		0.22166	0.19031	0.17658	0.17	0.00664	0.00250	0.00111	0.00062
			5		0.23706	0.19856	0.18083	0.17287	0.00870	0.00307	0.00128	0.00069
1.5	1.0	0.5	3	0.35355	0.40022	0.37993	0.36494	0.36298	0.00561	0.00273	0.00125	0.00092
			4		0.41838	0.38986	0.3702	0.36653	0.00728	0.00324	0.00137	0.00098
			5		0.43559	0.39952	0.37538	0.37004	0.00949	0.00393	0.00154	0.00107
1.5	1.5	0.5	3	0.51976	0.56065	0.54038	0.53019	0.52726	0.00455	0.00214	0.00122	0.00078
			4		0.57679	0.54946	0.53501	0.53053	0.00574	0.00247	0.00130	0.00082
			5		0.59182	0.5582	0.53973	0.53375	0.00735	0.00295	0.00143	0.00088
θ	λ	t	v		Chi-squared distribution ($P_2(\theta \mid t)$)							
1.5	0.5	0.5	3	0.15851	0.16023	0.1582	0.16018	0.15898	0.00286	0.00154	0.00080	0.00049
			5		0.14986	0.15273	0.15734	0.15707	0.00275	0.00152	0.00079	0.00049
			7		0.14019	0.14745	0.15455	0.15518	0.00284	0.00156	0.00079	0.00049
1.5	1.0	0.5	3	0.35355	0.34344	0.35031	0.34965	0.35276	0.00458	0.00235	0.00122	0.00087
			5		0.33011	0.34331	0.34607	0.35035	0.00498	0.00243	0.00126	0.00088
			7		0.31732	0.33646	0.34252	0.34796	0.00570	0.00261	0.00132	0.00090
1.5	1.5	0.5	3	0.51976	0.50816	0.51276	0.51605	0.51778	0.00438	0.00214	0.00125	0.00078
			5		0.49551	0.50619	0.51271	0.51554	0.00492	0.00230	0.00129	0.00079
			7		0.48319	0.49971	0.50939	0.51331	0.00575	0.00255	0.00136	0.00082
θ	λ	t	λ_1		Erlang-2 type distribution ($P_3(\theta \mid t)$)							
1.5	0.5	0.5	2	0.15851	0.17852	0.16786	0.16513	0.16232	0.00310	0.00158	0.00083	0.00050
			4		0.20941	0.1843	0.1736	0.16803	0.00513	0.00212	0.00100	0.00058
			6		0.23929	0.20051	0.18201	0.17372	0.00886	0.00317	0.00132	0.00071
1.5	1.0	0.5	2	0.35355	0.36737	0.36264	0.356	0.35698	0.00401	0.00223	0.00116	0.00086
			4		0.40474	0.38278	0.36659	0.36411	0.00572	0.00278	0.00126	0.00092
			6		0.43841	0.40184	0.37689	0.37109	0.00972	0.00406	0.00158	0.00109
1.5	1.5	0.5	2	0.51976	0.5308	0.52443	0.52195	0.52172	0.00355	0.00190	0.00118	0.00075
			4		0.5648	0.54307	0.53171	0.52831	0.00459	0.00215	0.00122	0.00078
			6		0.59434	0.56037	0.5411	0.53473	0.00751	0.00304	0.00145	0.00089

5. Discussion

In this paper, we study maximum likelihood (ML) method and the Bayesian estimation for the shape parameter (θ), reliability function (R(t)) of the Kumaraswamy distribution (under the exponential distribution, Chi-squared distribution and Erlang-2 type distribution as informative priors). We derive Bayes estimators under the squared error loss function with different priors. Finally, according to the simulation study we determine the best estimation method, we use the Mean Square Error (MSE) on each of the estimation obtained from methods. So our criterion is the best method that gives the smallest value of (MSE).

In general, simulation results shown that the estimator of θ has increase mean square errors (MSE) when the true value of θ and λ are increased in all cases,and the estimator for R(t) has decreased mean square errors (MSE) when the true value of θ and λ are increased in all cases except the case of θ =1.5. So we see from

- **Tables 4.1 show that the Bayes estimator of θ when the prior distribution for θ is exponential distribution has the smallest estimated MSE's compared with ML's ,and the maximum likelihood (ML) of θ has the smallest estimated MSE's compared with the Bayes estimator of θ when the prior distribution of θ is Chi-squared distribution. Also, we see that the Bayes estimator of θ when the prior distribution of θ is Erlang-2 type distribution with certain chosen set values (λ_1=4,6) has the smallest estimated MSE's compared with ML's.**
- **Tables 4.2 show that the Bayes estimator of θ when the prior distribution of θ is exponential distribution has the smallest estimated MSE's compared with ML's ,and the ML of θ has the smallest estimated MSE's compared with the Bayes estimator of θ when the prior distribution of θ is Chi-squared distribution .Also, we see that the Bayes estimator of θ when the prior distribution for θ is Erlang-2 type distribution have the smallest estimated MSE's compared with ML's .**
- **Table 4.3 and shows that the Bayes estimator of θ when the prior distribution of θ is exponential distribution with certain chosen set values (a=3,4) has the smallest estimated MSE's compared with ML's ,and the ML of θ has the smallest estimated MSE's compared with the Bayes estimator of θ when the prior distribution of θ is Chi-squared distribution. Also, we see that the Bayes estimator of θ when the prior distribution of θ is Erlang-2 type distribution with certain chosen set values (λ_1=2,4) has the smallest estimated MSE's compared with ML's.**
- **Tables 4.4 show that the Bayes estimator for R(t) when the prior distribution of θ is exponential distribution has the smallest estimated MSE's compared with MLE's .And the MLE for R(t) has the smallest estimated MSE's compared with the Bayes estimator for R(t) when the prior distribution of θ is Chi-squared distribution .Also, we see that the Bayes estimator for R(t) when the prior**

distribution for θ is Erlang-2 type distribution has the smallest estimated MSE's compared with MLE's .

- **Table 4.5 shows that the Bayes estimator for R(t) when the prior distribution for θ is exponential distribution has the smallest estimated MSE's compared with ML's .And the ML for R(t) has the smallest estimated MSE's compared with the Bayes estimator when the prior distribution of θ is Chi-squared distribution except for the case(θ=1, λ=0.5with v=3,5) have the smallest estimated MSE's compared with ML's .Also, we see the that Bayes estimator for R(t) when the prior distribution of θ is Erlang-2 type distribution with certain chosen set values (λ_1=2,4) has the smallest estimated MSE's compared with MLE's.**
- **Table 4.6 shows that the MLE for R(t) has the smallest estimated MSE's compared with the Bayes estimator for R(t) when the prior distribution of θ is exponential distribution .Also, we see the that Bayes estimator for R(t) when the prior distribution of θ is Chi-squared distribution with certain chosen set values (v=3,5,7) when θ =1.5, λ=0.5 has the smallest estimated MSE's compared with MLE's .And we see the that Bayes estimator for R(t) when the prior distribution of θ is Chi-squared distribution with certain chosen set values (v=3) when θ =1.5,λ=1,1.5 has the smallest estimated MSE's compared with MLE's .Also, we see the that Bayes estimator for R(t) when the prior distribution of θ is Erlang-2 type distribution with certain chosen set values (λ_1=2) when θ =1.5, λ=0.5,1 ,1.5 has the smallest estimated MSE's compared with MLE's .**

6.Conclusion

In general, simulation results shown that the Bayes estimator of θ when the prior distribution for θ is exponential distribution has the smallest estimated MSE's compared with ML's, and the maximum likelihood (ML) of θ has the smallest estimated MSE's compared with the Bayes estimator of θ when the prior distribution of θ is Chi-squared distribution .Also, simulation results showed that Bayes estimator of θ when the prior distribution of θ is Erlang-2 type distribution for certain chosen set values of (λ_1=4,6) when the true value of θ is 0.5 has the smallest estimated MSE's compared with ML's .Also, the same thing for (λ_1=2,4) when the true value θ is 1.5.

The maximum likelihood (ML) for R(t) has the smallest estimated MSE's compared with the Bayes estimator when the prior distribution of θ is exponential distribution and Chi-squared distribution for certain chosen set values of (v=3,5) when the true value of θ =1.0, λ=0.5, also for (v=3) when the true value of θ =1.0, λ=1. Also for certain chosen set values of (v=3) when the true value of θ =1.0, λ=1, and the maximum likelihood (ML) for R(t) has the smallest estimated MSE's compared with Bayes estimator when the prior distribution of θ is Erlang-2 type distribution for certain chosen set values of (λ_1=2,4) when the true value of θ is 1 for all λ .

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Appendix-A: The posterior distribution using three types of the prior distribution

1-The posterior distribution using exponential distribution as prior:

To find the posterior distribution using exponential distribution, we follow these steps:

When the prior distribution θ is exponential distribution, then the pdf is given by:

$$P_1(\theta) = a \exp(-a\theta), \text{ for } a,\theta > 0 \qquad \dots (A.1)$$

Then the posterior distribution of θ for the given the data $(t_1, t_2, \dots t_n)$ is given by:

$$P(\theta \backslash t) = \frac{L(\theta)P(\theta)}{\int_\theta L(\theta)P(\theta)\,d\theta} \qquad \dots (A.2)$$

Substituting the equation (4) and (A.1) in equation (A.2), we get:

$$P_1(\theta \backslash t) = \frac{[\theta^n \lambda^n \prod_{i=1}^n t^{\lambda-1} \prod_{i=1}^n (1-t^\lambda)^{\theta-1}][an \exp(-a\theta)]}{\int_\theta [\theta^n \lambda^n \prod_{i=1}^n t^{\lambda-1} \prod_{i=1}^n (1-t^\lambda)^{\theta-1}][a \exp(-a\theta)]\,d\theta} \qquad \dots (A.3)$$

$$P_1(\theta \backslash t) = \frac{\theta^n \exp(-\theta(a-\sum_{i=1}^n \log(1-t_i^\lambda)))}{\int_\theta \theta^n \exp(-\theta(a-\sum_{i=1}^n \log(1-t_i^\lambda)))]d\theta} \qquad \dots (A.4)$$

By multiplying the integral in equation (A.4) by the quantity which equals to

$$[\Gamma(n+1) / (a-\sum_{i=1}^n \log(1-t_i^\lambda))^{n+}][(a-\sum_{i=1}^n \log(1-t_i^\lambda))^{n+1} / \Gamma(n+1)]$$

where $\Gamma(.)$ is a gamma function .Rewrite the equation (A.4) as follow, then we get,

$$P_1(\theta \backslash t) = \frac{1}{\Gamma(n+1)A_1(\theta,t)} (a-\sum_{i=1}^n \log(1-t_i^\lambda))^{n+1} \theta^{(n+1)-1} \exp(-\theta(a-\sum_{i=1}^n \log(1-t_i^\lambda))) ,$$

for $a,\theta,\lambda,n > 0$ …(A.5)

Where $A_1(\theta,t)$ equals to

$$A_1(\theta,t) = \int_0^\infty \frac{1}{\Gamma(n+1)} (a-\sum_{i=1}^n \log(1-t_i^\lambda))^{n+1} \theta^{(n+1)-1} \exp(-\theta(a-\sum_{i=1}^n \log(1-t_i^\lambda)))\,d\theta = 1.$$

Be the integral of the pdf of the gamma distribution. We get the posterior distribution of θ given the data $(t_1, t_2, \dots t_n)$ as follows:-

$$P_1(\theta\backslash t)= \frac{1}{\Gamma(n+1)} (a-\sum_{i=1}^{n} \log(1- t_i^{\lambda}))^{n+1} \theta^{(n+1)-1} \exp(-\theta(a-\sum_{i=1}^{n} \log(1- t_i^{\lambda})),\text{for}$$

$a, \theta, \lambda, n > 0$ …(A.6)

It means that $P_1(\theta\backslash t)$ ~gamma distribution with new parameters

$(a_{(new)}=n+1, b_{(new)}= (a-\sum_{i=1}^{n} \log(1- t_i^{\lambda})))$.

2-<u>The posterior distribution using chi-square distribution as prior:</u>

To find the posterior distribution using chi-squared distribution, we follow these steps:

When the prior distribution θ is chi-squared distribution, then the pdf is given by:

$$P_2(\theta)=\frac{1}{\Gamma(\frac{v}{2})}(\frac{1}{2})^{(1/2)} \theta^{(v/2)-1}\exp(-\frac{1}{2}\theta) \text{ , for } v,\theta>0 \qquad \text{…(A.7)}$$

Then the posterior distribution of θ for the given the data $(t_1, t_2, \ldots t_n)$ is given by:

$$P(\theta\backslash t)=\frac{L(\theta)P(\theta)}{\int_{\theta} L(\theta)P(\theta)\,d\theta} \qquad \text{…(A.2)}$$

Substituting the equation (4) and (A.1) in equation (A.2), we get:

$$P_2(\theta\backslash t)=\frac{[\theta^n \lambda^n \prod_{i=1}^{n} t^{\lambda-1} \prod_{i=1}^{n}(1-t^{\lambda})^{\theta-1}][\frac{1}{\Gamma(\frac{v}{2})}(\frac{1}{2})^{(1/2)} \theta^{(v/2)-1}\exp(-\frac{1}{2}\theta)]}{\int_{\theta}[\theta^n \lambda^n \prod_{i=1}^{n} t^{\lambda-1} \prod_{i=1}^{n}(1-t^{\lambda})^{\theta-1}][\frac{1}{\Gamma(\frac{v}{2})}(\frac{1}{2})^{(1/2)} \theta^{(v/2)-1}\exp(-\frac{1}{2}\theta)]d\theta} \qquad \text{…(A.8)}$$

$$P_2(\theta\backslash t)=\frac{\theta^{(n+0.5v)-1} \exp(-\theta(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}))}{\int_{\theta} \theta^{(n+0.5v)-1} \exp(-\theta(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}))]d\theta} \qquad \text{…(A.9)}$$

By multiplying the integral in equation (A.9) by the quantity which equals to

$$[\Gamma(n+0.5v)/(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}))^{n+0.5v}]\ [(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}))^{n+0.5v}/\Gamma(n+0.5v)]$$

where $\Gamma(.)$ is a gamma function .Rewrite the equation (A.9) as follow, then we get,

$$P_2(\theta\backslash t) = \frac{1}{\Gamma(n+0.5v)\,A_2(\theta,t)} \left(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)^{n+(0.5v)} \theta^{(n+0.5v)-1} \exp\left(-\theta\left(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right) \quad \ldots(A.10)$$

Where $A_2(\theta,t)$ equals to

$$A_2(\theta,t) = \int_0^{\infty} \frac{1}{\Gamma(n+0.5v)} \left(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)^{n+(0.5v)} \theta^{(n+0.5v)-1} \exp\left(-\theta\left(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right) d\theta = 1.$$

Be the integral of the pdf of gamma distribution. We get the posterior distribution of θ given the data $(t_1, t_2, \ldots t_n)$ as follows:-

$$P_2(\theta\backslash t) = \frac{1}{\Gamma(n+0.5v)} \left(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)^{n+(0.5v)} \theta^{(n+0.5v)-1} \exp\left(-\theta\left(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right), \quad \text{for } v,\theta,\lambda, n > 0 \quad \ldots(A.11)$$

It means that $P_2(\theta\backslash t)$ ~gamma distribution with new parameters

$$\left(a_{(new)} = n+0.5v,\ b_{(new)} = \left(0.5-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right).$$

3-<u>The posterior distribution using erlang-2 distribution as prior:</u>

To find the posterior distribution using erlang-2 distribution, we follow these steps:

When the prior distribution θ is erlang distribution, then the pdf is given by:

$$P_3(\theta) = \lambda_1^2\ \theta \exp(-\lambda_1 \theta) \quad \text{for } \lambda_1, \theta > 0 \quad \ldots(A.12)$$

Then the posterior distribution of θ for the given the data $(t_1, t_2, \ldots t_n)$ is given by:

$$P(\theta\backslash t) = \frac{L(\theta)P(\theta)}{\int_\theta L(\theta)P(\theta)\,d\theta} \quad \ldots(A.2)$$

Substituting the equation (4) and (A.7) in equation (A.2), we get:

$$P_3(\theta\backslash t) = \frac{\left[\theta^n \lambda^n \prod_{i=1}^{n} t^{\lambda-1} \prod_{i=1}^{n} (1-t^{\lambda})^{\theta-1}\right]\left[\lambda_1^2\ \theta \exp(-\lambda_1)\right]}{\int_\theta \left[\theta^n \lambda^n \prod_{i=1}^{n} t^{\lambda-1} \prod_{i=1}^{n} (1-t^{\lambda})^{\theta-1}\right]\left[\lambda_1^2\ \theta \exp(-\lambda_1)\right] d\theta} \quad \ldots(A.13)$$

$$P_3(\theta\backslash t) = \frac{\theta^{n+1} \exp\left(-\theta\left(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right)}{\int_\theta \theta^{n+1} \exp\left(-\theta\left(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right) d\theta} \quad \ldots(A.14)$$

By multiplying the integral in equation (A.9) by the quantity which equals to

$$[(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))^{n+2}\ /\Gamma(n+2)]\ [\Gamma(n+2)\ /(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))^{n+2}\ ]$$

where $\Gamma(.)$ is a gamma function .Rewrite the equation (A.14) as follow, then we get,

$$P_3(\theta\backslash t)=\frac{1}{\Gamma(n+2)A_3(\theta,t)}(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))^{n+2}\theta^{(n+2)-1}\exp(-\theta(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))$$

…(A.15)

Where $A_3(\theta,t)$ equals to

$$A_3(\theta,t)=\int_0^{\infty}\frac{1}{\Gamma(n+2)}(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))^{n+2}\ \theta^{(n+2)-1}\ \exp(-\theta(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))\,d\theta=1.$$

Be the integral of the pdf of the gamma distribution. Then we get the posterior distribution of θ given the data $(t_1, t_2, \dots t_n)$ as follows:-

$$P_3(\theta\backslash t)=\frac{1}{\Gamma(n+2)}\ (\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))^{n+2}\ \theta^{(n+2)-1}\ \exp(-\theta(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))\ ,\ \text{for}\ \lambda_1,\theta,\lambda,n>0\quad \dots (A.16)$$

It means that $P_3(\theta\backslash t)$ ~gamma distribution with new parameters

$$(a_{(new)}=n+2,\ b_{(new)}=(\lambda_1-\sum_{i=1}^{n}\log(1-t_i^{\lambda}))).$$

Appendix-B

The following is the derivation of Bayes 'estimators under the squared error loss function.

First: Bayes 'Estimators for the shape parameter (θ)

-The squared error loss function

By using squared error loss function $L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2$, the risk function is

$$R(\hat{\theta},\theta)=\int_{\theta}(\hat{\theta}-\theta)^2\ P(\theta\backslash t)\,d\theta \qquad \dots (B.1)$$

$$R(\hat{\theta},\theta)=\hat{\theta}^2\int_{\theta}P(\theta\backslash t)\,d\theta-2\hat{\theta}\int_{\theta}\theta\,P(\theta\backslash t)\,d\theta+\int_{\theta}\theta^2P(\theta\backslash t)\,d\theta \quad \dots (B.2)$$

$$R(\hat{\theta},\theta)=\hat{\theta}^2-2\hat{\theta}E(\theta\backslash t)+E(\theta^2\backslash t) \qquad \dots (B.3)$$

Let $(\partial/\partial\hat{\theta})\ R(\hat{\theta},\theta)=0$, we get Bayes estimator of θ denoted by $\hat{\theta}_{SE}$ for the above prior as follows

$$\hat{\theta}_{SEi}=E(\theta\backslash t)=\int_{\forall\theta}\theta\,P_i(\theta\backslash t)\,d\theta \qquad i=1,2,3 \qquad \dots (B.4)$$

1. Bayes estimation using exponential distribution as prior:

To obtain the Bayes' estimator under exponential distribution as prior. Substituting the equation (A.6) in equation (B.4), we get:

$$\hat{\theta}_{SE1} = E(\theta \backslash t) = \int_0^\infty \theta\, P_1(\theta \backslash t)\, d\theta \qquad \text{for } i=1 \qquad \ldots(B.4)$$

$$\hat{\theta}_{SE1} = \int_0^\infty \theta \frac{1}{\Gamma(n+1)} \left(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+1} \theta^{(n+1)-1} \exp\left(-\theta\left(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)\right) d\theta$$

.Be the mean of the gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE1} = (n+1) \Big/ \left(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right) \qquad \text{for } a, \theta, \lambda, n > 0 \qquad \ldots(B.5)$$

2. Bayes estimation using chi-square distribution as prior:

To obtain the Bayes' estimator under chi-square distribution as prior. Substituting the equation (A.11) in equation (B.4), we get:

$$\hat{\theta}_{SE2} = E(\theta \backslash t) = \int_0^\infty \theta\, P_2(\theta \backslash t)\, d\theta \qquad \text{for } i=2 \qquad \ldots(B.4)$$

$$\hat{\theta}_{SE2} = \int_0^\infty \theta \frac{1}{\Gamma(n+0.5v)} \left(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+(0.5v)} \theta^{n+(0.5v)-1} \exp\left(-\theta\left(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)\right) d\theta$$

. Be the mean of the gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE2} = (n+0.5v) \Big/ \left(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right) \qquad \text{for } v, \theta, \lambda, n > 0 \qquad \ldots(B.6)$$

3. Bayes estimation using erlang-2 distribution as prior:

To obtain the Bayes' estimator under erlang-2_distribution as prior. Substituting the equation (A.16) in equation (B.4), we get:

$$\hat{\theta}_{SE3} = E(\theta \backslash t) = \int_0^\infty \theta\, P_3(\theta \backslash t)\, d\theta \qquad \text{for } i=3 \qquad \ldots(B.4)$$

$$\hat{\theta}_{SE3} = \int_0^\infty \theta \frac{1}{\Gamma(n+2)} \left(\lambda_1-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+2} \theta^{(n+2)-1} \exp\left(-\theta\left(\lambda_1-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)\right) d\theta.$$

Be the mean of the gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE3} = (n+2) \Big/ \left(\lambda_1-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right) \qquad \text{, for } \lambda_1, \theta, \lambda, n > 0 \qquad \ldots(B.7)$$

Second: Bayes 'Estimators for the Reliability function

-The squared error loss function

By using squared error loss function $L(\hat{R},R)=(\hat{R}-R)^2$,the risk function in

$$R(\hat{R},R)=\int_{\theta}(\hat{R}-R)^2\ P(\theta\backslash t)\,d\theta \qquad \ldots(B.8)$$

$$R(\hat{R},R)=\hat{R}^2\int_{\theta}P(\theta\backslash t)\,d\theta-2\hat{R}\int_{\theta}R\,P(\theta\backslash t)\,d\theta+\int_{\theta}R^2P(\theta\backslash t)\,d\theta \qquad \ldots(B.9)$$

$$R(\hat{R},R)=\hat{R}^2-2\hat{R}\,E(R\backslash t)+E(R^2\backslash t) \qquad \ldots(B.10)$$

Let $(\partial/\partial\hat{R})\,R(\hat{R},R)=0$, we get Bayes estimator of the Reliability function $(R(t))$ denoted by $\hat{R}_{SE}(t)$ for the above prior as follows

$$\hat{R}_{SEi}(t)=E(R\backslash t)=\int_0^\infty R(t)\,P_i(\theta\backslash t)\,d\theta \qquad i=1,2,3 \qquad \ldots(B.11)$$

1. Bayes estimation using exponential distribution as prior:

To obtain the Bayes' estimator under exponential distribution as prior. Substituting the equation (A.6) in equation (B.11), we get:

$$\hat{R}_{SEi}(t)=E(R\backslash t)=\int_0^\infty R(t)\,P_i(\theta\backslash t)\,d\theta \qquad i=1 \qquad \ldots(B.12)$$

Where R (t) is the Reliability function as follow

$R_T(t)=(1-t^{\lambda})^{\theta}$, we can rewrite it as follow

$$R_T(t)=\exp[\theta\log(1-t^{\lambda})] \qquad \ldots(B.13)$$

$$\hat{R}_{SE1}(t)=\int_0^\infty \exp[\theta\log(1-t^{\lambda})]\frac{1}{\Gamma(n+1)}\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)^{n+1}\theta^{(n+1)-1}\exp\left(-\theta\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right)d\theta \qquad \ldots(B.14)$$

$$\hat{R}_{SE1}(t)=\int_0^\infty \frac{1}{\Gamma(n+1)}\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)^{n+1}\theta^{(n+1)-1}\exp\left[-\theta\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})-\log(1-t^{\lambda})\right)\right]d\theta \qquad \ldots(B.15)$$

By multiplying the integral in equation (B.15) by the quantity which equals to

$$\left(\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})-\log(1-t^{\lambda})\right)\right)^{n+1}/\left(\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})-\log(1-t^{\lambda})\right)\right)^{n+1}$$

Then we have

$$\hat{R}_{SE1}(t)=\left(\left[\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})\right)\right]^{n+1}/\left[\left(a-\sum_{i=1}^{n}\log(1-t_i^{\lambda})-\log(1-t^{\lambda})\right)\right]^{n+1}\right)B_1(\theta,t) \qquad \ldots(B.16)$$

Where $B_1(\theta,t)$ equals to

$$B_1(\theta,t)=\int_0^\infty \frac{1}{\Gamma(n+1)} \left((a-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right)^{n+1} \theta^{(n+1)-1} \exp\left(-\theta(a-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right) d\theta=1.$$

Be the integral of the pdf of the gamma distribution. Then we get the Bayes estimator of R(t) as the following formula:

$$\hat{R}_{SE1}(t) = \left((a-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+1} / \left((a-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right)^{n+1}, \text{ for } a,\theta,\lambda,n>0 \quad \ldots(B.17)$$

2. <u>Bayes estimation using chi-square distribution as prior:</u>

To obtain the Bayes' estimator under chi-square distribution as prior. Substituting the equation (A.11) in equation (B.11), we get:

$$\hat{R}_{SEi}(t) = E(R \backslash t) = \int_0^\infty R(t)\ P_i(\theta \backslash t)\, d\theta \qquad i=2 \qquad \ldots(B.12)$$

Where R (t) is the Reliability function as in equation (B.13)

$$\hat{R}_{SE2}(t) = \int_0^\infty \exp[\theta \log(1-t^{\lambda})] \frac{1}{\Gamma(n+0.5v)} \left(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+(0.5v)} \theta^{(n+0.5v)-1}$$

$$\exp\left(-\theta(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right) d\theta \qquad \ldots(B.18)$$

$$\hat{R}_{SE2}(t) = \int_0^\infty \frac{1}{\Gamma(n+0.5v)} \left(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right)^{n+(0.5v)} \theta^{(n+0.5v)-1}$$

$$\text{Exp}\left(-\theta(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right)) d\theta \qquad \ldots(B.19)$$

By multiplying the integral in equation (B.19) by the quantity which equals to

$$\left((0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right)^{n+0.5v} / \left((0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right)^{n+0.5v}$$

Then we have

$$\hat{R}_{SE2}(t) = \left(((0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda})\right))^{n+0.5v} / \left((0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right)^{n+0.5v} B_2(\theta,t) \quad \ldots(B.20)$$

Where $B_2(\theta,t)$ equals to

$$B_2(\theta,t)=\int_0^\infty \frac{1}{\Gamma(n+0.5v)} \left((0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right)^{n+1} \theta^{(n+1)-1} \exp\left(-\theta(0.5-\sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})\right) d\theta=1.$$

Be the integral of the pdf of the gamma distribution. Then we get the Bayes estimator of R(t) as the following formula:

$\hat{R}_{SE2}$ **(t) =((0.5-** $\sum_{i=1}^{n}$ **log (1– t$_i^{\lambda}$))$^{n+0.5v}$ /((0.5-** $\sum_{i=1}^{n}$ **log (1– t$_i^{\lambda}$)– log (1–t$^{\lambda}$))$^{n+0.5v}$) for v,θ , λ, n > 0 … (B.21)**

3. Bayes estimation using erlang-2 distribution as prior:

To obtain the Bayes' estimator under erlang-2_distribution as prior. Substituting the equation (A.16) in equation (B.4), we get:

$$\hat{R}_{SEi}(t) = E(R\backslash t) = \int_0^{\infty} R(t)\ P_i(\theta\backslash t)\, d\theta \qquad i=2 \qquad \ldots(B.12)$$

Where R (t) is the Reliability function as in equation (B.13)

$$\hat{R}_{SE3}(t) = \int_0^{\infty} \exp[\theta \log(1-t^{\lambda})] \frac{1}{\Gamma(n+2)} (\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}))^{n+2}\ \theta^{(n+2)-1}$$

$$\exp(-\theta(\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda})))\, d\theta \qquad \ldots(B.22)$$

$$\hat{R}_{SE3}(t) = \int_0^{\infty} \frac{1}{\Gamma(n+2)} (\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}))^{n+2}\ \theta^{(n+2)-1}$$

$$\exp(-\theta((\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})))\, d\theta \qquad \ldots(B.23)$$

By multiplying the integral in equation (B.23) by the quantity which equals to

$((\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda}))^{n+2} / ((\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda}))^{n+2})$

Then we have

$$\hat{R}_{SE3}(t) = ((\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}))^{n+2} / ((\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda}))^{n+2} B_3(\theta,t) \qquad \ldots(B.24)$$

Where $B_3(\theta,t)$ equals to

$$B_3(\theta,t) = \int_0^{\infty} \frac{1}{\Gamma(n+2)} (\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda})) - \log(1-t^{\lambda})]^{n+1}\ \theta^{(n+2)-1}\ \exp(-\theta(\lambda_1 - \sum_{i=1}^{n} \log(1-t_i^{\lambda}) - \log(1-t^{\lambda})))\, d\theta = 1.$$

Be the integral of the pdf of the gamma distribution. Then we get the Bayes estimator of R(t) as the following formula:

$\hat{R}_{SE3}$ **(t) =((λ$_1$-** $\sum_{i=1}^{n}$ **log (1– t$_i^{\lambda}$))$^{n+2}$ /((λ$_1$-** $\sum_{i=1}^{n}$ **log (1– t$_i^{\lambda}$)– log (1–t$^{\lambda}$))$^{n+2}$), for λ$_1$,θ, λ, n > 0 …(B.25)**

Appendix-C

The following is the derivation of moments 'estimators for the parameters for prior distributions with pdf as given in table -1 .

1. Method of moments for the exponential distribution

Let $\theta_1, \theta_2, \ldots, \theta_n$ are identically and independently distributed random samples taken from the exponential distribution with pdf:

$$P_1(\theta) = a \exp(-a\theta) \quad , \text{ for } a,\theta > 0$$

The estimate ($\hat{a}$) of the parameter (a) using method of moments we have

$$M_r = m_r \text{ for } r=1, 2, \ldots \qquad \ldots \text{ (C.1)}$$

So the first moment about origin is $M_1 = E(\theta) = (1/a)$, i.e the mean the exponential distribution, and the first sample moment about origin is

$m_1 = \sum_{i=1}^{n} \theta_i / n = \bar{\theta}$ .So, using the method of moments we have, $M_1 = m_1$. Further simplifying we have $1/\hat{a} = \sum_{i=1}^{n} \theta_i / n$. Consequently, the moments estimate for the rate parameter is

$$\hat{a} = n / \sum_{i=1}^{n} \theta_i \qquad \ldots \text{ (C.2)}$$

2. Method of moments for the Chi-squared distribution

Let $\theta_1, \theta_2, \ldots, \theta_n$ are identically and independently distributed random samples taken from the Chi-squared distribution with pdf:

$$P_2(\theta) = \frac{1}{\Gamma(\frac{v}{2})} (\frac{1}{2})^{(1/2)} \theta^{(v/2)-1} \exp(-\frac{1}{2}\theta) \quad , \text{ for } v,\theta > 0$$

The estimate ($\hat{v}$) of the parameter (v) using method of moments we have

$$M_r = m_r \text{ for } r=1, 2, \ldots \qquad \ldots \text{ (C.1)}$$

So the first moment about the origin is $M_1 = E(\theta) = v$, i.e the mean the Chi-squared distribution, and the first sample moment about the origin is

$m_1 = \sum_{i=1}^{n} \theta_i / n = \bar{\theta}$ .So, using the method of moments we have, $M_1 = m_1$. Further simplifying we have $\hat{v} = \sum_{i=1}^{n} \theta_i / n$. Consequently, the moments estimate for the parameter is

$$\hat{v} = \sum_{i=1}^{n} \theta_i / n = \bar{\theta} \ .$$

3. Method of moments for the Erlang-2 type distribution

Let $\theta_1, \theta_2, \ldots, \theta_n$ are identically and independently distributed random samples taken from the Erlang-2 type distribution with pdf:

$$P_3(\theta) = \lambda_1^2 \ \theta \exp(-\lambda_1 \theta) \quad , \text{ for } \lambda_1,\theta > 0$$

The estimate $\hat{\lambda}_1$ of the parameter (λ_1) using method of moments we have
$M_r = m_r$ for r=1, 2, … ... (C.1)
So the first moment about the origin is $M_1 = E(\theta) = 2/\lambda_1$, i.e the mean the Erlang-2 type distribution, and the first sample moment about the origin is $m_1 = \sum_{i=1}^{n} \theta_i / n = \bar{\theta}$ .So, using the method of moments we have, $M_1 = m_1$. Further simplifying we have $2/\hat{\lambda}_1 = \sum_{i=1}^{n} \theta_i / n$. Consequently, the moments estimate for the parameter is

$$\hat{\lambda}_1 = 2/\bar{\theta} = 2n / \sum_{i=1}^{n} \theta_i \quad \text{... (C.4)}$$

مقارنة مقدرات معلمة الشكل ودالة المعولية لتوزيع Kumaraswamy

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الخلاصة

في هذا البحث، تمت دراسة طريقة الإمكان الأعظم (MLE)وطريقة بيز لتقدير معلمة الشكل (θ) ودالة المعولية ((R(t لتوزيع Kumaraswamy (باستعمال التوزيع الاسي، وتوزيع مربع كاي وتوزيع ارلنك من النوع الثاني كتوزيعات اولية)، بالإضافة الى ذلك استعملنا طريقة العزوم لتقدير معلمات التوزيعات الأولية. وتم اشتقاق مقدرات بيز باستعمال دالة الخسارة التربيعية. واستعملنا أسلوب المحاكاة لمقارنة أداء كل مقدر. فقد تم توليد بيانات لعدة قيم لمعلمة الشكل (θ) من توزيع Kumaraswamy ولأحجام مختلفة من العينات (الصغيرة والمتوسطة والكبيرة). تبين من نتائج المحاكاة بان طريقة بيز هي الأفضل وفقا لأقل قيم لمتوسط مربعات الخطاء (MSE) ولكل احجام العينات.

المصطلحات الرئيسة للبحث / **توزيع Kumaraswamy , معلمة الشكل , دالة المعولية, الإمكان الاعظم ,طريقة بيز , التوزيعات الأولية (التوزيع الاسي , توزيع مربع كاي , توزيع ارلنك),دالة الخسارة التربيعية.**