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Bayes Analysis for the Scale Parameter of Gompertz Distribution

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Abstract

In this paper, we investigate the behavior of the bayes estimators, for the scale parameter of the Gompertz distribution under two different loss functions such as, the squared error loss function, the exponential loss function (proposed), based different double prior distributions represented as erlang with inverse levy prior, erlang with non-informative prior, inverse levy with non-informative prior and erlang with chi-square prior.

The simulation method was fulfilled to obtain the results, including the estimated values and the mean square error (MSE) for the scale parameter of the Gompertz distribution, for different cases for the scale parameter of the Gompertz distribution, with different samples sizes. The estimates have been compared in terms of their mean-squared error (MSE).

The results of this paper show that bayes estimators of the scale parameter (ω) of the Gompertz distribution, under the exponential loss function (proposed) are superior to the bayes estimators (ω) under the squared error loss function, based on erlang-chi-square double prior with $(k=1, b=2)$ for all samples sizes and for all the true values of (ω), in terms of their mean-squared error (MSE).

Paper type: Research paper.

Keywords: The Gompertz distribution, bayes estimation, the square error loss function, the exponential error loss function.

1. Introduction

The Gompertz distribution is commonly used in many applied problems, particularly in lifetime data analysis. Benzamin Gompertz (1825) [3] applied the Gompertz distribution in the human mortality and actuarial, and has been applied in various fields especially in reliability and life testing studies, actuarial science, epidemiological and biomedical studies. Also, it is used in computer Science and Marketing Science. The Gompertz distribution has an exponentially increasing failure rate for the life of the systems and is often used to model highly negatively skewed data for the analysis of survival.

Many authors have contributed to the studies of statistical methodology and characterization of this distribution, and investigate the effects on Bayes estimation for the parameters of Gompertz distribution based on different loss functions, and prior distributions which represented by informative prior and non-informative prior, we can mention some of them as:

Saracoglu et.al (2009) [8] studied different estimation procedures for the reliability, R, when X and Y were two independent but not identically Gompertz random variables, which were represented by the maximum likelihood estimator, and the uniformly minimum variance unbiased estimator and the bayes estimator based on the mean squared error loss function. They depended on simulation study to compare the different estimators of R. They noted that the mean square error of bayes was smaller than the mean square error for maximum likelihood, and the minimum variance unbiased, except for small sample size.

Wang et.al (2016) [9] studied the statistical inference for the Gompertz distribution under record values. They used the maximum likelihood and bayes estimates for the parameters of the Gompertz distribution in addition to some reliability performances such as survival and hazard rate functions. Also, they developed the exact confidence interval/ region and approximate confidence intervals for the Gompertz parameters.

Mohie El-Din and Sharawy (2017) [7] discussed estimation for the parameters of the Gompertz distribution (GD) based on general progressively type-II right censored order statistics. They derived bayes estimators based on squared error loss function, Al-Bayyati loss function, linear-exponential loss function, generalization of the entropy loss function, they applied these types in two cases, first when the shape parameter was known and the second was unknown shape and scale parameter. They performed a simulation study to investigate the behavior of the bayes estimators. they concluded that the Al-Bayyati loss function was better than others loss functions. Also, they concluded that Al-Bayyati loss function and linear-exponential loss function were better than others loss functions.

Dey et.al (2018) [2] delt with the different methods of estimation of the unknown parameters of Gompertz distribution. They derived the maximum likelihood estimator and bayes estimators and posterior risk under different loss functions of the unknown parameters of Gompertz distribution. Also, they studied various mathematical and statistical properties of the Gompertz distribution (such as quantiles, moments, moment generating function, hazard rate, mean residual lifetime, mean past lifetime, stochasic ordering, stressstrength parameter, various entropies, Bonferroni and Lorenz curves and order statistics).

Moala and Dey (2018) [6] used maximum likelihood estimation to derive the approximate confidence intervals of the parameters of the Gompertz distribution. Also, they derived Bayes estimators of unknown parameter of the Gompertz distribution under different priors, they supposed that the parameters had the independent gamma prior distributions, Jeffrey 's prior, maximal data information prior (MDIP), singpurwalla's prior and elicited prior. They used a simulation study to investigate the behavior of the proposed methods. They depended on the average biases and the mean squared errors to compare the Bayes estimators with different priors and MLE.

Ieren et. al (2019) [4] discussed another extension of the Gompertz distribution using the power transformation approach to introduce a three-parameter probability distribution which was known as Power Gompertz distribution(PGD). They derived, studied and discussed some properties of the PGD. They estimated the three parameters of the new model using the method of maximum likelihood estimation. They used a real life dataset, its descriptive statistics, graphical summary and applications. They compared the fits of the Power Gompertz Distribution (PGD) and Gompertz Distribution (GD) using a dataset on the remission times of a random sample of 128 bladder cancer patients.

Lee and Seo (2020) [5] proposed different approaches based on the weighted regression framework and pivotal quantity to estimate unknown parameters of the Gompertz distribution under the progressive Type-II censoring scheme. They performed Simulation Study to evaluation and comparison, the mean squared errors (MSEs) and biases of the provided estimators. They have proved that the pivot-based estimators were superior to the MLEs and weighted least-square estimators in terms of the MSE and bias.

The objective of this study is to derive the bayes estimators for a scale parameter of Gompertz distribution. The Posterior distributions of scale parameter are derived under erlang- inverse levy prior, erlang-non-informative prior, inverse levy -non informative prior, and erlang-chi-square prior distributions. Bayes estimators are derived by using two loss functions represented by square error loss function and exponential error loss function under the three double priors and posterior distributions respectively. The performance of these estimators have assessed on the basis of their mean square errors (MSE).

2.The Gompertz distribution

The random variable x follows the Gompertz distribution with the shape and scale parameters as $\gamma > 0$ and $\omega > 0$ respectively, if it has the following probability density function (pdf) [1]:

$$f(x;\gamma,\omega) = \omega\, e^{\gamma x}\, e^{-(\frac{\omega}{\gamma})(\exp(\gamma x)-1)} \quad , \quad x > 0\,, \quad \gamma > 0,\ \omega > 0 \qquad (1)$$

The corresponding (cdf) of the Gompertz distribution is given by

$$F(x;\gamma,\omega) = 1 - e^{-(\frac{\omega}{\gamma})(\exp(\gamma x)-1)} \quad , \quad x > 0\,, \quad \gamma > 0,\ \omega > 0 \qquad (2)$$

for the shape parameter equals to $\gamma = 1$, the probability density function of the distribution reduces to [5,7].

$$f(x;\omega) = \omega\, e^{x}\, e^{-\omega(\exp(x)-1)} \quad , \quad x > 0\,, \quad \omega > 0 \qquad (3)$$

And the cumulative distribution function will be as the following

$$F(x;\ \omega) = 1 - e^{-\omega(\exp(x)-1)} \quad , \quad x > 0 \ , \quad \omega > 0 \qquad (4)$$

3. Bayesian Estimation

The Bayes estimators for the unknown parameter ω of the Gompertz distribution have been derived under the squared error loss function and the exponential error loss function based on different double priors. The respective expressions have been presented in the following:

3.1 Posterior distribution

Posterior distribution for the unknown parameter ω of the Gompertz distribution based on the below considerations has been derived.

(a). For erlang- inverse levy distributions

We choose the random variables for ω to follow erlang distribution with hyper parameter (k) [13]as

$$\upsilon_1(\omega) = k^2 \ \omega \ \exp(-k\,\omega) \qquad \text{with} \quad \omega, k > 0 \qquad (5)$$

and inverse levy distribution with hyper parameter (v) [11]as

$$\upsilon_2(\omega) = \sqrt{\frac{v}{2\pi}} \ \omega^{-\frac{1}{2}} \ \exp(-\frac{v}{2}\ \omega) \qquad \text{with} \quad \omega, v > 0 \qquad (6)$$

be independent random variables for ω,then the double prior for the parameter ω , and their density functions will be $\upsilon_{12}(\omega) = \upsilon_1(\omega) \times \upsilon_2(\omega)$, i.e.

$$\upsilon_{12}(\omega) = k^2 \ \omega \ \exp(-k\,\omega) \times \sqrt{\frac{v}{2\pi}} \ \omega^{-\frac{1}{2}} \ \exp(-\frac{v}{2}\ \omega)$$

$$\upsilon_{12}(\omega) = (k^2 \sqrt{\frac{v}{2\pi}}) \quad \omega^{\frac{1}{2}} \ \exp(-\omega(\frac{v}{2} + k)) \qquad (7)$$

Then, the posterior distribution of ω is given by [1] :

$$q_1(\omega \backslash x) = \frac{\ell(\omega \backslash x_1, x_2, \ldots, x_n)\ \upsilon_{12}(\omega)}{\int_{\omega} \ell(\omega \backslash x_1, x_2, \ldots, x_n)\ \upsilon_{12}(\omega) d\omega} \qquad (8)$$

Where $\ell(\omega \backslash x_1, x_2, \ldots, x_n)$ be the likelihood function for the $(x_1, x_2, \ldots, x_n)$ observations is defined by [1]:

$$\ell(\omega \backslash x) = \prod_{i=1}^{n} f(x_i;\omega) = \omega^n e^{\sum_{i=1}^{n} x_i} e^{-\omega \sum_{i=1}^{n} (\exp(x_i)-1)} \qquad (9)$$

Substituting "Eqn (9)" and "Eqn (7)" in "Eqn (8)", yields the posterior probability density function of the shape parameter ω as the following:

$$q_1(\omega \backslash x) = \frac{\omega^n e^{\sum_{i=1}^{n} x_i} e^{-\omega \sum_{i=1}^{n} (\exp(x_i)-1)[(k^2 \sqrt{\frac{v}{2\pi}}) \quad \omega^{\frac{1}{2}} \ \exp(-\omega(\frac{v}{2}+k))]}}{\int_0^{\infty} \omega^n e^{\sum_{i=1}^{n} x_i} e^{-\omega \sum_{i=1}^{n} (\exp(x_i)-1)\ [(k^2 \sqrt{\frac{v}{2\pi}}) \quad \omega^{\frac{1}{2}} \ \exp(-\omega(\frac{v}{2}+k))]} d\omega}$$

$$q_1(\omega \backslash x) = \frac{\omega^{n+0.5}\, e^{-\omega(s+k+0.5v)}}{\int_0^\infty \omega^{n+0.5}\, e^{-\omega(s+k+0.5v)}\, d\omega} \qquad (10)$$

Where $s = \sum_{i=1}^{n} (\exp(x_i) - 1)$ **.Rewrite** $\omega^{n+0.5} = \omega^{(n+1.5)-1}$ **and by multiplying the integral in "Eqn (10)" by the quantity which is equal to**

$\left(\frac{(s+k+0.5v)^{(n+1.5)}}{\Gamma(n+1.5)}\right)\left(\frac{\Gamma(n+1.5)}{(s+k+0.5v)^{(n+1.5)}}\right)$ **,where** $\Gamma(.)$ **is a gamma function , it yields**

$$q_1(\omega \backslash x) = \frac{(s+k+0.5v)^{(n+1.5)}}{\Gamma(n+1.5)A(x;\omega)}\, \omega^{(n+1.5)-1}\, e^{-\omega(s+k+0.5v)}$$

,Where

$A(x;\omega) = \int_0^\infty \frac{(s+k+0.5v)^{(n+1.5)}}{\Gamma(n+1.5)}\, \omega^{(n+1.5)-1}\, e^{-\omega(s+k+0.5v)}\, d\omega = 1$ **.be the integral of the pdf of gamma distribution [7]. The posterior distribution of** (ω) **is gamma distribution [10] as**

$$q_1(\omega \backslash x) = \frac{(s+k+0.5v)^{(n+1.5)}}{\Gamma(n+1.5v)}\, \omega^{(n+1.5)-1}\, e^{-\omega(s+k+0.5v)} \quad , \quad \omega > 0\ , k, v, n > 0 \qquad (11)$$

i.e. $(\omega \backslash x)$ gamma$((n+1.5), (s+k+0.5v))$ **with posterior mean is** $E(w \backslash x) = \frac{(n+1.5)}{(s+k+0.5v)}$ **and posterior variance is** $\text{var}(w \backslash x) = \frac{(n+1.5)}{(s+k+0.5v)^2}$ **.**

(b). For erlang distribution-non-informative prior

We used erlang distribution with hyper parameter (k) **as defined in "Eqn (5)", and non-informative with hyper parameter** (c_1) **as follows**

$$\upsilon_3(\omega) = \frac{1}{\omega^{c_1}} \qquad \omega\ , c_1 > 0 \qquad (12)$$

be independent random variables for ω **,then the double prior for the parameter** ω**, and their density functions will be** $\upsilon_{13}(\omega) = \upsilon_1(\omega) \times \upsilon_3(\omega)$ **,i.e.**

$$\upsilon_{13}(\omega) = k^2\ \omega\ \exp(-k\omega) \times \omega^{-c_1}$$

$$\upsilon_{13}(\omega) = k^2\, \omega^{1-c_1} \exp(-k\omega) \qquad (13)$$

Then, the posterior distribution of ω **is given by substituting "Eqn (9)" and "Eqn (13)" in "Eqn (8)", yields the posterior probability density function of the shape parameter** ω **as the following:**

$$q_2(\omega \backslash x) = \frac{\omega^n\, e^{\sum_{i=1}^n x_i}\, e^{-\omega \sum_{i=1}^n (\exp(x_i)-1)} [k^2\, \omega^{1-c_1} \exp(-k\omega)]}{\int_0^\infty \omega^n\, e^{\sum_{i=1}^n x_i}\, e^{-\omega \sum_{i=1}^n (\exp(x_i)-1)}\, [k^2\, \omega^{1-c_1} \exp(-k\omega)]\, d\omega}$$

$$q_2(\omega \backslash x)=\frac{\omega^{n+1-c_1}\ e^{-\omega(s+k)}}{\int_0^\infty \omega^{n+1-c_1}\ e^{-\omega(s+k)}\ d\omega} \qquad (14)$$

Where $s=\sum_{i=1}^{n}(\exp(x_i)-1)$ **.Rewrite** $\omega^{n+1-c_1}=\omega^{(n+2-c_1)-1}$ **and by multiplying the integral in "Eqn (14)" by the quantity which is equal to** $\left(\frac{(s+k)^{(n+2-c_1)}}{\Gamma(n+2-c_1)}\right)\left(\frac{\Gamma(n+2-c_1)}{(s+k)^{(n+2-c_1)}}\right)$ **,where** $\Gamma(.)$ **is a gamma function, it yields**

$$q_2(\omega \backslash x)=\frac{(s+k)^{(n+2-c_1)}}{\Gamma(n+2-c_1)A1(x;\omega)}\omega^{(n+2-c_1)-1}\ e^{-\omega(s+k)}$$

,Where

$A1(x;\omega)=\int_0^\infty \frac{(s+k)^{(n+2-c_1)}}{\Gamma(n+2-c_1)}\omega^{(n+2-c_1)-1}\ e^{-\omega(s+k)}\ d\omega=1$ **,be the integral of the pdf of gamma distribution [7]. The posterior distribution of** (ω) **is gamma distribution [10] as**

$$q_2(\omega \backslash x)=\frac{(s+k)^{(n+2-c_1)}}{\Gamma(n+2-c_1)}\omega^{(n+2-c_1)-1}\ e^{-\omega(s+k)}\ ,\ \omega>0\ ,c_1,k,n>0 \qquad (15)$$

i.e. $(\omega \backslash x)\ \ gamma((n+2-c_1),(s+k))$ **with posterior mean is**

$E(w \backslash x)=\frac{(n+2-c_1)}{(s+k)}$ **and posterior variance is** $var(w \backslash x)=\frac{(n+2-c_1)}{(s+k)^2}$.

<u>(c). For inverse levy distribution-non-informative prior</u>

We used inverse levy distribution with hyper parameter (v) **as defined in "Eqn (6)",and non-informative with hyper parameter** (c_1) **as defined in "Eqn (12)", be independent random variables for** ω **,then the double prior for the parameter** ω**, and their density functions will be** $\upsilon_{23}(\omega)=\upsilon_2(\omega)\times\upsilon_3(\omega)$ **,i.e.**

$$\upsilon_{23}(\omega)=\sqrt{\frac{v}{2\pi}}\ \omega^{-\frac{1}{2}}\ \exp(-\frac{v}{2}\ \omega)\times\omega^{-c_1}$$

$$\upsilon_{23}(\omega)=\sqrt{\frac{v}{2\pi}}\ \omega^{-\frac{1}{2}-c_1}\exp(-\frac{v}{2}\ \omega) \qquad (16)$$

Then, the posterior distribution of ω **is given by substituting "Eqn (9)" and "Eqn (16)" in "Eqn (8)", yields the posterior probability density function of the shape parameter** ω **as the following:**

$$q_3(\omega \backslash x)=\frac{\omega^n\ e^{\sum_{i=1}^{n}x_i}\ e^{-\omega\sum_{i=1}^{n}(\exp(x_i)-1)}[\sqrt{\frac{v}{2\pi}}\ \omega^{-0.5-c_1}\exp(-\frac{v}{2}\ \omega)]}{\int_0^\infty \omega^n\ e^{\sum_{i=1}^{n}x_i}\ e^{-\omega\sum_{i=1}^{n}(\exp(x_i)-1)}[\sqrt{\frac{v}{2\pi}}\ \omega^{-0.5-c_1}\exp(-\frac{v}{2}\ \omega)]\ d\omega}$$

$$q_3(\omega\setminus x)=\frac{\omega^{n-0.5-c_1}\ e^{-\omega\sum_{i=1}^{n}(s+0.5v)}}{\int_0^{\infty}\omega^{n-0.5-c_1}\ e^{-\omega\sum_{i=1}^{n}(s+0.5v)}\,d\omega}\qquad(17)$$

Where $s=\sum_{i=1}^{n}(\exp(x_i)-1)$ **.Rewrite** $\omega^{n-0.5-c_1}=\omega^{(n+0.5-c_1)-1}$ **and by multiplying the integral in "Eqn (17)" by the quantity which is equals to**

$$\left(\frac{(s+0.5v)^{(n+0.5-c_1)}}{\Gamma(n+0.5-c_1)}\right)\left(\frac{\Gamma(n+0.5-c_1)}{(s+k)^{(n+0.5-c_1)}}\right)$$ **,where** $\Gamma(.)$ **is a gamma function , it yields**

$$q_3(\omega\setminus x)=\frac{(s+0.5v)^{(n+0.5-c_1)}}{\Gamma(n+0.5-c_1)A2(x;\omega)}\omega^{(n+0.5-c_1)-1}\ e^{-\omega(s+0.5v)}$$ **,Where**

$$A2(x;\omega)=\int_0^{\infty}\frac{(s+0.5v)^{(n+0.5-c_1)}}{\Gamma(n+0.5-c_1)}\omega^{(n+0.5-c_1)-1}\ e^{-\omega(s+0.5v)}\,d\omega=1,$$ **be the integral of the pdf of gamma distribution [7]. Then the posterior distribution of** (ω) **is gamma distribution [10] as**

$$q_3(\omega\setminus x)=\frac{(s+0.5v)^{(n+0.5-c_1)}}{\Gamma(n+0.5-c_1)}\omega^{(n+0.5-c_1)-1}\ e^{-\omega(s+0.5v)}\quad,\omega>0\ ,c_1,v,n>0\qquad(18)$$

i.e. $(\omega\setminus x)\ \text{gamma}((n+0.5-c_1),(s+0.5v))$ **with posterior mean is** $E(w\setminus x)=\dfrac{(n+0.5-c_1)}{(s+0.5v)}$ **and posterior variance is** $\text{var}(w\setminus x)=\dfrac{(n+0.5-c_1)}{(s+0.5v)^2}$**.**

<u>(d). For erlang- chi-square distributions</u>

We use erlang distribution with hyper parameter (k) **as defined in "Eqn (5)",and chi-square distribution with hyper parameter (b) [1,12] as**

$$\upsilon_4(\omega)=\frac{1}{2^{\frac{b}{2}}\sqrt{\frac{b}{2}}}\ \omega^{\frac{b}{2}-1}\ \exp(-\frac{1}{2}\ \omega)\quad\text{with}\quad\omega,b>0\qquad(19)$$

be independent random variables for ω **,then the double prior for the parameter** ω **, and their density functions will be** $\upsilon_{14}(\omega)=\upsilon_1(\omega)\times\upsilon_4(\omega)$**, i.e.**

$$\upsilon_{14}(\omega)=k^2\ \omega\ \exp(-k\omega)\times\frac{1}{2^{\frac{b}{2}}\sqrt{\frac{b}{2}}}\ \omega^{\frac{b}{2}-1}\ \exp(-\frac{1}{2}\ \omega)$$

$$\upsilon_{14}(\omega)=(k^2\frac{1}{2^{\frac{b}{2}}\sqrt{\frac{b}{2}}})\ \omega^{\frac{b}{2}}\exp(-\omega(k+0.5)\qquad(20)$$

Then, the posterior distribution of ω is given by substituting "Eqn (9)" and "Eqn (20)" in "Eqn (8)", yields the posterior probability density function of the shape parameter ω as the following:

$$q_4(\omega \backslash x) = \frac{\omega^n e^{\sum_{i=1}^n x_i} e^{-\omega \sum_{i=1}^n (\exp(x_i)-1)[(k^2 \frac{1}{2^{\frac{b}{2}} \sqrt{\frac{b}{2}}}) \times \omega^{\frac{b}{2}} \exp(-\omega(k+0.5))]}}{\int_0^\infty \omega^n e^{\sum_{i=1}^n x_i} e^{-\omega \sum_{i=1}^n (\exp(x_i)-1)[(k^2 \frac{1}{2^{\frac{b}{2}} \sqrt{\frac{b}{2}}}) \times \omega^{\frac{b}{2}} \exp(-\omega(k+0.5))]} d\omega}$$

$$q_4(\omega \backslash x) = \frac{\omega^{n+0.5b} e^{-\omega(s+k+0.5)}}{\int_0^\infty \omega^{n+0.5b} e^{-\omega(s+k+0.5)} d\omega} \qquad (21)$$

Where $s = \sum_{i=1}^n (\exp(x_i)-1)$ **.Rewrite** $\omega^{n+0.5b} = \omega^{(n+0.5b+1)-1}$ **and by multiplying the integral in "Eqn (21)" by the quantity which is equals to**

$$\left(\frac{(s+k+0.5)^{(n+0.5b+1)}}{\Gamma(n+0.5b+1)}\right)\left(\frac{\Gamma(n+0.5b+1)}{(s+k+0.5)^{(n+0.5b+1)}}\right)$$ **,where** $\Gamma(.)$ **is a gamma function , it yields**

$$q_4(\omega \backslash x) = \frac{(s+k+0.5)^{(n+0.5b+1)}}{\Gamma(n+0.5b+1) A3(x;\omega)} \omega^{(n+0.5b+1)-1} e^{-\omega(s+k+0.5)}$$ **,Where**

$$A3(x;\omega) = \int_0^\infty \frac{(s+k+0.5)^{(n+0.5b+1)}}{\Gamma(n+0.5b+1)} \omega^{(n+0.5b+1)-1} e^{-\omega(s+k+0.5)} d\omega = 1$$ **,be the integral of the pdf of gamma distribution [7]. The posterior distribution of (ω) is gamma distribution [10] as**

$$q_4(\omega \backslash x) = \frac{(s+k+0.5)^{(n+0.5b+1)}}{\Gamma(n+0.5b+1)} \omega^{(n+0.5b+1)-1} e^{-\omega(s+k+0.5)}, \ \omega > 0, k, b, n > 0 \quad (22)$$

i.e. $(\omega \backslash x) \ \text{gamma}((n+0.5b+1),(s+k+0.5))$ **with posterior mean is** $E(w \backslash x) = \frac{(n+0.5b+1)}{(s+k+0.5)}$ **and posterior variance is** $\text{var}(w \backslash x) = \frac{(n+0.5b+1)}{(s+k+0.5)^2}$.

3.2 Bayes Estimation under Square Error Loss Function

Bayes estimators and posterior risk for ω under different double priors which are used to derive the posterior distributions as in section 3.1.We can derive bayes estimators using the square error loss function based on the posterior distributions. The risk function has been defined as

$$R_1(\hat{\omega}, \omega) = E[L_1(\hat{\omega} - \omega)^2], \ R_1(\hat{\omega} - \omega) = \hat{\omega}^2 - 2\hat{\omega} E(\omega \backslash x) + E(\omega^2 \backslash x).$$

The value of ω minimizes the risk function under square error loss function which satisfies the following condition $\frac{\partial}{\partial \hat{\omega}} R(\hat{\omega} - \omega) = 0$,we have bayes estimators of ω denoted by $\hat{\omega}$

$$\hat{\omega} = E(\omega \backslash x) = \int_0^\infty \omega\, q(\omega \backslash x)\, d\omega \qquad (23)$$

i.e., $\hat{\omega} = E(\omega \backslash x)$ is equal to the posterior mean for different double priors informative and non-informative priors as derived in section 3.1.

3.3 Bayes Estimation under the exponential Loss Function

Bayes estimator and posterior risk for ω under different double priors such as defined in previous section using the exponential loss function (proposed). Then, the risk function has been defined as

$$R_2(\hat{\omega}, \omega) = E[L_2(\exp(\hat{\omega}) - \exp\omega)^2],$$

$R_2(\hat{\omega}, \omega) = \exp(2\hat{\omega}) - 2\exp(\hat{\omega})E(\exp(\omega) \backslash t) + E(\exp(2\omega) \backslash t)$ **.The value of ω minimizes the risk function under the proposed loss function which satisfies the following condition $\frac{\partial}{\partial \hat{\omega}} R(\hat{\omega} - \omega) = 0$, we have bayes estimator of ω denoted by $\hat{\omega}$ for the above prior as follows:**

$$\hat{\omega} = \ln E(\exp(\omega) \backslash x) = \ln\left(\int_0^\infty \exp(\omega)\, q(\omega \backslash x)\, d\omega\right) \qquad (24)$$

The bayes estimators and corresponding risks under other double priors can be derived in the similar manner.

(A). For erlang- inverse levy distributions

Substituting "Eqn (11)" in "Eqn (24)", yields bayes estimator of the shape parameter ω as the following:

We have $\hat{\omega} = \ln E(\exp(\omega) \backslash x) = \ln\left(\int_0^\infty \exp(\omega)\, q_1(\omega \backslash x)\, d\omega\right) \qquad (24)$

$$\hat{\omega} = \ln\left(\int_0^\infty \exp(w) \frac{(s+k+0.5v)^{(n+1.5)}}{\Gamma(n+1.5v)} \omega^{(n+1.5)-1} e^{-\omega(s+k+0.5v)}\, d\omega\right)$$

$$\hat{\omega} = \ln\left(\int_0^\infty \frac{(s+k+0.5v)^{(n+1.5)}}{\Gamma(n+1.5v)} \omega^{(n+1.5)-1} e^{-\omega(s+k+0.5v-1)}\, d\omega\right) \qquad (25)$$

By multiplying the integral in "Eqn (25)" by the quantity which is equals to $\frac{(s+k+0.5v-1)^{(n+1.5)}}{(s+k+0.5v-1)^{(n+1.5)}}$ **.,it yield** $\hat{\omega} = \ln\left(\frac{(s+k+0.5v)^{(n+1.5)}}{(s+k+0.5v-1)^{(n+1.5)}} B(x, \omega)\right)$ **where**

$B(x,\omega)=\int_0^\infty \frac{(s+k+0.5v-1)^{(n+1.5)}}{\Gamma(n+1.5v)}\omega^{(n+1.5)-1}e^{-\omega(s+k+0.5v-1)}d\omega=1$ **, be the integral of the pdf of gamma distribution [10], i.e.**

$$\hat{\omega}=\ln(\frac{s+k+0.5v}{s+k+0.5v-1})^{(n+1.5)} \qquad (26)$$

(B). For erlang distribution-non-informative prior

Substituting "Eqn (15)" in "Eqn (24)", yields bayes estimator of the shape parameter ω as the following:

We have $\hat{\omega}=\ln E(\exp(\omega)\backslash x)=\ln(\int_0^\infty \exp(\omega)q_2(\omega\backslash x)d\omega) \qquad (24)$

$$\hat{\omega}=\ln(\int_0^\infty \exp(\omega)\frac{(s+k)^{(n+2-c_1)}}{\Gamma(n+2-c_1)}\omega^{(n+2-c_1)-1}e^{-\omega(s+k)}d\omega)$$

$$\hat{\omega}=\ln(\int_0^\infty \frac{(s+k)^{(n+2-c_1)}}{\Gamma(n+2-c_1)}\omega^{(n+2-c_1)-1}e^{-\omega(s+k-1)}d\omega) \qquad (27)$$

By multiplying the integral in "Eqn (27)" by the quantity which is equals to $\frac{(s+k-1)^{(n+2-c_1)}}{(s+k-1)^{(n+2=c_1)}}$ **,it yield**

$\hat{\omega}=\ln(\frac{(s+k)^{(n+2-c_1)}}{(s+k-1)^{(n+2-c_1)}}B1(x,\omega))$ **where**

$B1(x,\omega)=\int_0^\infty \frac{(s+k-1)^{(n+2-c_1)}}{\Gamma(n+2-c_1)}\omega^{(n+2-c_1)-1}e^{-\omega(s+k-1)}d\omega=1$ **, be the integral of the pdf of gamma distribution [10], i.e.**

$$\hat{\omega}=\ln(\frac{s+k}{s+k-1})^{(n+2-c_1)} \qquad (28)$$

(C). For inverse levy distribution-non-informative prior

Substituting "Eqn (18)" in "Eqn (24)", yields bayes estimator of the shape parameter ω as the following:

We have $\hat{\omega}=\ln E(\exp(\omega)\backslash x)=\ln(\int_0^\infty \exp(\omega)q_3(\omega\backslash x)d\omega) \qquad (24)$

$$\hat{\omega}=\ln(\int_0^\infty \exp(\omega)\frac{(s+0.5v)^{(n+0.5-c_1)}}{\Gamma(n+0.5-c_1)}\omega^{(n+0.5-c_1)-1}e^{-\omega(s+0.5v)}d\omega)$$

$$\hat{\omega}=\ln(\int_0^\infty \frac{(s+0.5v)^{(n+0.5-c_1)}}{\Gamma(n+0.5-c_1)}\omega^{(n+0.5-c_1)-1}e^{-\omega(s+0.5v-1)}d\omega) \qquad (29)$$

By multiplying the integral in "Eqn (29)" by the quantity which is equals to $\frac{(s+0.5v-1)^{(n+0.5-c_1)}}{(s+0.5v-1)^{(n+0.5-c_1)}}$ **,it yield** $\hat{\omega}=\ln(\frac{(s+0.5v)^{(n+0.5-c_1)}}{(s+0.5v-1)^{(n+0.5-c_1)}}B2(x,\omega))$ **, where** $B2(x,\omega)=\int_0^\infty \frac{(s+0.5v)^{(n+0.5-c_1)}}{\Gamma(n+0.5-c_1)}\omega^{(n+0.5-c_1)-1}e^{-\omega(s+0.5v-1)}d\omega=1$ **be the integral of the pdf of gamma distribution [10], i.e.**

$$\hat{\omega}=\ln(\frac{s+0.5v}{s+0.5v-1})^{(n+0.5-c_1)} \qquad (30)$$

(D). For erlang- chi-square distributions

Substituting "Eqn (22)" in "Eqn (24)", yields bayes estimator of the shape parameter ω **as the following:**

We have $\hat{\omega}=\ln E(\exp(\omega)\backslash x)=\ln(\int_0^\infty \exp(\omega)\,q_4(\omega\backslash x)\,d\omega) \qquad (24)$

$$\hat{\omega}=\ln(\int_0^\infty \exp(\omega)\frac{(s+k+0.5)^{(n+0.5b+1)}}{\Gamma(n+0.5b+1)}\omega^{(n+0.5b+1)-1}e^{-\omega(s+k+0.5)}d\omega)$$

$$\hat{\omega}=\ln(\int_0^\infty \frac{(s+k+0.5)^{(n+0.5b+1)}}{\Gamma(n+0.5b+1)}\omega^{(n+0.5b+1)-1}e^{-\omega(s+k-0.5)}d\omega) \qquad (31)$$

By multiplying the integral in "Eqn (31)" by the quantity which is equals to $\frac{(s+k-0.5)^{(n+0.5b+1)}}{(s+k-0.5)^{(n+0.5b+1)}}$ **,it yield** $\hat{\omega}=\ln(\frac{(s+k+0.5)^{(n+0.5b+1)}}{(s+k-0.5)^{(n+0.5b+1)}}B3(x,\omega))$ **, where** $B3(x,\omega)=\int_0^\infty \frac{(s+k-0.5)^{(n+0.5b+1)}}{\Gamma(n+0.5b+1)}\omega^{(n+0.5b+1)-1}e^{-\omega(s+k-0.5)}d\omega=1$ **be the integral of the pdf of gamma distribution [10], i.e.**

$$\hat{\omega}=\ln(\frac{s+k+0.5}{s+k-0.5})^{(n+0.5b+1)} \qquad (32)$$

4. Simulation Study

We used simulation method by using MATLAB-R2018a program ,under 5000 replications is considered , to generate random samples of sizes n = (15, 25, 50, 100) from Gompertz distribution using the quantile function from "Eqn (4)" as $x_i=\ln(1-\frac{1}{\omega}\ln(1-F_i))$ **, where** $F_i=U_i$ **is a uniform distribution with (0,1) , for different cases for the scale parameter of the Gompertz model have been represented by** $\omega=0.02, 0.5, 1, 2$ **,for small and medium and large values of scale parameter of the Gompertz model ,with selected different values for the hyper parameters** k, v, c_1**, and b are known of the double prior distributions as**

- **For erlang- inverse levy prior with** $(k=0.5, v=2)$**.**

- **For erlang-non informative prior with** $(k=1, c_1=2)$.
- **For inverse levy -non informative prior with** $(\nu=1, c_1=1)$.
- **For erlang-chi-square prior with** $(k=1, b=2)$.

To investigate the behavior of the proposed methods and their estimates, the estimates have been compared in terms of their mean-square error (MSE) which has been computed as

$$MSE = \frac{1}{5000}\sum_{r=1}^{5000}(\hat{\omega}(r)-\omega)^2 \quad (33)$$

The results are presented in tables (1 and 2), including the estimated values $(\hat{\omega})$ **and the mean square error (MSE) for parameter** (ω) **of the Gompertz distribution under square error loss function based on different double prior. Also, the results are presented in tables (3 and 4), including the estimated values** $(\hat{\omega})$ **and the mean square error (MSE) for parameter** (ω) **of the Gompertz distribution under exponential error loss function based on different double prior.**

Table .1 Estimated values $(\hat{\omega})$ **of parameter** (ω) **for the Gompertz distribution using square error loss function under different double prior.**

N	Method	ω = 0.02	0.5	1	2
15	Erlang- inverse levy prior with $(k=0.5, \nu=2)$	0.023462	0.55452	1.0581	1.9211
	erlang-non-informative prior with $(k=1, c_1=2)$	0.022769	0.54753	1.0624	1.9843
	Inverse levy -non informative prior with $(\nu=1, c_1=1)$	0.02065	0.50545	0.99826	1.9247
	Erlang-chi-square prior with $(k=1, b=2)$	0.024173	0.57133	1.0902	1.9793
25	Erlang- inverse levy prior with $(k=0.5, \nu=2)$	0.02203	0.53498	1.0349	1.9475
	Erlang-non-informative prior with $(k=1, c_1=2)$	0.021624	0.53047	1.0364	1.9863
	Inverse levy -non informative prior with $(\nu=1, c_1=1)$	0.020385	0.50523	0.99735	1.949
	Erlang-chi-square prior with $(k=1, b=2)$	0.022445	0.54508	1.0545	1.9843
50	Erlang- inverse levy prior with $(k=0.5, \nu=2)$	0.020979	0.51914	1.0196	1.9773
	Erlang-non-informative prior with $(k=1, c_1=2)$	0.02078	0.51676	1.02	1.9971
	Inverse levy -non informative prior with $(\nu=1, c_1=1)$	0.020173	0.50417	1.0002	1.9779
	Erlang-chi-square prior with $(k=1, b=2)$	0.021183	0.52418	1.0295	1.9965
100	Erlang- inverse levy prior with $(k=0.5, \nu=2)$	0.020501	0.50869	1.0098	1.9904
	Erlang-non-informative prior with $(k=1, c_1=2)$	0.020402	0.50746	1.0099	2.0004
	Inverse levy -non informative prior with $(\nu=1, c_1=1)$	0.020101	0.5012	0.99995	1.9906
	Erlang-chi-square prior with $(k=1, b=2)$	0.020602	0.51119	1.0148	2.0002

Table .2 The mean square error of parameter (ω) for the Gompertz distribution using square error loss function under different double prior.

N	Method	ω 0.02	0.5	1	2
15	Erlang- inverse levy prior with $(k=0.5, v=2)$	5.520e-5	0.023007	0.074019	0.18229
	Erlang-non-informative prior with $(k=1, c_1=2)$	4.844e-5	0.022616	0.080907	0.21575
	Inverse levy -non informative prior with $(v=1, c_1=1)$	3.402e-5	0.018132	0.073893	0.24151
	Erlang-chi-square prior with $(k=1, b=2)$	6.329e-5	0.026354	0.083121	0.18733
25	Erlang- inverse levy prior with $(k=0.5, v=2)$	2.461e-5	0.012946	0.042834	0.13074
	Erlang-non-informative prior with $(k=1, c_1=2)$	2.240e-5	0.01272	0.044955	0.14468
	Inverse levy -non informative prior with $(v=1, c_1=1)$	1.773e-5	0.010975	0.04229	0.15422
	Erlang-chi-square prior with $(k=1, b=2)$	2.725e-5	0.014201	0.046164	0.13311
50	Erlang- inverse levy prior with $(k=0.5, v=2)$	9.653e-6	0.005839	0.020086	0.072291
	Erlang-non-informative prior with $(k=1, c_1=2)$	9.142e-6	0.005762	0.020534	0.076303
	Inverse levy -non informative prior with $(v=1, c_1=1)$	8.076e-6	0.005292	0.019776	0.078534
	Erlang-chi-square prior with $(k=1, b=2)$	1.026e-5	0.006164	0.020956	0.073187
100	Erlang- inverse levy prior with $(k=0.5, v=2)$	4.534e-6	0.002699	0.010266	0.038432
	Erlang-non-informative prior with $(k=1, c_1=2)$	4.404e-6	0.002680	0.010375	0.039514
	Inverse levy -non informative prior with $(v=1, c_1=1)$	4.129e-6	0.002575	0.01018	0.040021
	Erlang-chi-square prior with $(k=1, b=2)$	4.687e-6	0.002774	0.010489	0.038718

Note: The shadow cells represent the smallest value of RMSE.

Table .3 Estimated values $(\hat{\omega})$ of parameter (ω) for the Gompertz distribution using exponential error loss function under different double prior.

N	Method	ω 0.02	0.5	1	2
15	Erlang- inverse levy prior with $(k=0.5, v=2)$	0.02348	0.56471	1.0961	2.0494
	Erlang-non-informative prior with $(k=1, c_1=2)$	0.022786	0.55781	1.1021	2.1276
	Inverse levy -non informative prior with $(v=1, c_1=1)$	0.021378	0.53291	1.073	2.1477
	Erlang-chi-square prior with $(k=1, b=2)$	0.021346	0.51337	0.99641	1.8631
25	Erlang- inverse levy prior with $(k=0.5, v=2)$	0.022039	0.54069	1.0565	2.0256
	Erlang-non-informative prior with $(k=1, c_1=2)$	0.021633	0.53619	1.0586	2.0696
	Inverse levy -non informative prior with $(v=1, c_1=1)$	0.02081	0.52117	1.04	2.0761
	Erlang-chi-square prior with $(k=1, b=2)$	0.020792	0.51008	0.99673	1.911
50	Erlang- inverse levy prior with $(k=0.5, v=2)$	0.020983	0.52183	1.03	2.017
	Erlang-non-informative prior with $(k=1, c_1=2)$	0.020784	0.51945	1.0305	2.0381
	Inverse levy -non informative prior with $(v=1, c_1=1)$	0.020381	0.51192	1.0209	2.0397
	Erlang-chi-square prior with $(k=1, b=2)$	0.020372	0.50663	1	1.9582
100	Erlang- inverse levy prior with $(k=0.5, v=2)$	0.020503	0.50998	1.0149	2.0104
	Erlang-non-informative prior with $(k=1, c_1=2)$	0.020404	0.50876	1.015	2.0207
	Inverse levy -non informative prior with $(v=1, c_1=1)$	0.020204	0.505	1.0101	2.0211
	Erlang-chi-square prior with $(k=1, b=2)$	0.0202	0.50244	0.99991	1.9807

Table .4 The mean square error of parameter (ω) for the Gompertz distribution using exponential error loss function under different double prior.

N	Method	ω 0.02	0.5	1	2
15	Erlang- inverse levy prior with $(k=0.5, v=2)$	5.547e-5	0.025854	0.091386	0.23393
	Erlang-non-informative prior with $(k=1, c_1=2)$	4.867e-5	0.025402	0.1007	0.30695
	Inverse levy -non informative prior with $(v=1, c_1=1)$	3.798e-5	0.022124	0.098848	0.37349
	Erlang-chi-square prior with $(k=1, b=2)$	3.765e-5	0.018085	0.067913	0.21005
25	Erlang- inverse levy prior with $(k=0.5, v=2)$	2.469e-5	0.013908	0.048561	0.15128
	Erlang-non-informative prior with $(k=1, c_1=2)$	2.246e-5	0.013643	0.051099	0.17625
	Inverse levy -non informative prior with $(v=1, c_1=1)$	1.899e-5	0.012378	0.049816	0.19475
	Erlang-chi-square prior with $(k=1, b=2)$	1.890e-5	0.011007	0.040386	0.14198
50	Erlang- inverse levy prior with $(k=0.5, v=2)$	9.669e-6	0.006066	0.021438	0.078151
	Erlang-non-informative prior with $(k=1, c_1=2)$	9.156e-6	0.005978	0.02193	0.084369
	Inverse levy -non informative prior with $(v=1, c_1=1)$	8.361e-6	0.005641	0.021486	0.088286
	Erlang-chi-square prior with $(k=1, b=2)$	8.341e-6	0.005313	0.019357	0.075139
100	Erlang- inverse levy prior with $(k=0.5, v=2)$	4.537e-6	0.002750	0.010602	0.040022
	Erlang-non-informative prior with $(k=1, c_1=2)$	4.407e-6	0.002728	0.010716	0.041582
	Inverse levy -non informative prior with $(v=1, c_1=1)$	4.204e-6	0.002651	0.010599	0.042473
	Erlang-chi-square prior with $(k=1, b=2)$	4.199e-6	0.002578	0.010075	0.039117

Note: The shadow cells represent the smallest value of RMSE.

5. Discussion

The results of the simulation and comparison show that in table.2, the bayes estimators of the scale parameter (ω) for the Gompertz distribution using square error loss function under

- **Inverse levy -non informative double prior with $(v=1, c_1=1)$, for all samples sizes ,when the true values of $\omega = 0.02, 0.5, 1$.**
- **Erlang- inverse levy double prior with $(k=0.5, v=2)$, for all samples sizes ,when the true values of $\omega = 2$**

have less than mean square error compared with the other bayes estimators. For the results which are listed in table.4, we see that the bayes estimators using the exponential error loss function under the double prior erlang-chi-square with $(k=1, b=2)$ have less than mean squared error compared with the other estimators, for all samples sizes and for all the true values of (ω).

6. Conclusion

In this study, we have used two double priors for comparison. In double prior we have more information than single prior.

We have derived the posterior distributions for the unknown parameter ω of the Gompertz distribution, and provided the bayes estimators of the scale parameter (ω) using square error loss function and exponential error loss function under the different double priors which are erlang- inverse levy , erlang-non-informative ,inverse levy -non informative and erlang-chi-square. On the basis of the result of the the simulation study, the following conclusions can be drawn:

- **The estimates for all estimators converge to true value in all cases when the sample size increases.**
- **The mean square error of all estimators decreases when the sample size (n)increases.**
- **The mean square error of all estimators increases when the true value (ω) increases.**

For all samples sizes, using erlang-chi-square double prior with $(k=1, b=2)$ as the double prior ,we can see the exponential error loss function is better than square error loss functions.

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تحليل بيز لمعلمة القياس لتوزيع Gompertz

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مستخلص البحث

في هذا البحث، نتناول دراسة سلوك مقدرات بيز لمعلمة القياس لتوزيع **Gompertz** باستعمال دالتين خسارة متمثلة بدالة الخسارة التربيعية ودالة الخسارة الاسية مستنده الى توزيعات اولية مضاعفة .تحت افتراض وجود معلومتين مختلفة لتوزيع معلمة القياس لتوزيع **Gompertz** , كتوزيع ارلنك مع توزيع معكوس ليفي , توزيع ارلنك مع توزيع غير معلوماتي و توزيع معكوس ليفي مع توزيع غير معلوماتي , توزيع ارلنك مع توزيع مربع كاي .

استعملت المحاكاة للحصول على نتائج البحث متمثلة بالقيم المقدرة لمعلمة القياس لتوزيع **Gompertz** ومتوسط مربعات الخطا لمعلمة القياس لتوزيع **Gompertz** , باختيار عدة حالات مختلفة لمعلمة القياس لتوزيع **Gompertz** استعملت لتوليد البيانات ولأحجام مختلفة من العينات .اعتمد متوسط مربعات الخطاء لقياس دقة التقديرات بين طرق بيز لتقدير لمعلمة القياس لتوزيع **Gompertz**.

تبين نتائج البحث , بان مقدرات بيز لمعلمة القياس لتوزيع **Gompertz** باستعمال ودالة الخسارة الاسية (المقترحة) تفوقت على مقدرات بيز باستعمال دالة الخسارة التربيعية , تحت افتراض بان التوزيع الاولي المضاعف يكون توزيع ارلنك مع توزيع مربع كاي بالمعلمتين $(k=1, b=2)$, ولكل الحالات المفترضة لمعلمة القياس لتوزيع **Gompertz** و لكل أحجام العينات .

نوع البحث: ورقة بحثية.

المصطلحات الرئيسة للبحث: توزيع Gompertz، معلمة القياس ، تقدير بيز ، دالة الخسارة التربيعية، ودالة الخسارة الاسية.