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A Comparative Study for Estimate Fractional Parameter of ARFIMA Model

Ammar Muayad Saber
Department of Statistics
College of Administration and Economics
University of Baghdad, Iraq
ammar.m@coadec.uobaghdad.edu.iq

Rabab Abdulrida Saleh
Department of Statistics
College of Administration and Economics
University of Baghdad, Iraq
rabab.saleh@coadec.uobaghdad.edu.iq

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Abstract:

Long memory analysis is one of the most active areas in econometrics and time series where various methods have been introduced to identify and estimate the long memory parameter in partially integrated time series. One of the most common models used to represent time series that have a long memory is the ARFIMA (Auto Regressive Fractional Integration Moving Average Model) which diffs are a fractional number called the fractional parameter. To analyze and determine the ARFIMA model, the fractal parameter must be estimated. There are many methods for fractional parameter estimation. In this research, the estimation methods were divided into indirect methods, where the Hurst parameter is estimated first, and then the fractional integration parameter is estimated from it by a relation between them. As for direct methods, the fractional integration parameter is estimated directly without relying on Hurst's parameter, and most of them are semi parametric methods. In this paper, some of the most common direct methods were used to estimate the fraction modulus namely (Geweke-Porter-Hudak, Smoothed Geweke-Porter-Hudak, Local Whittle, Wavelet and weighted wavelet), using simulation method with different value of (d) and different size of time series. The comparison between the methods was done using the mean squared error (MSE). It turns out that the best methods to estimate the fractional parameter is (Local Whittle).

The ARFIMA model was generated by a function programmed by the MATLAB statistical program.

Keywords: **Time series, Hurst exponent, ARFIMA model, Differences, Fractional integration, Wavelet transformation, and Estimating long memory.**

1. Introduction:

Time series with long memory can be observed in many areas of application which has attracted lots of interest in statistics and many applications.

The estimation of long memory (d) in the fractionally integrated process has been inspected widely in the literatures and different estimation methods will be introduced.

In 1980, the researchers Granger and Joyeux put forward the idea of the fractional integration parameter (d) in terms of the integral being fractional number, which arises from Box Jenkins generalization of (p, d, q) models.

In 1981, the researcher Hosking defined the time series of the type (ARFIMA), which is an extended case of the time series of the type (ARIMA), and the differences can be taken as fractional values. The researcher also defined the factor of fractional differences in the form of an indeterminate binomial series in the back word- Shift operator; also, he reached the mathematical formulas for the autocorrelation functions and the covariances of the fractional integration operations and proved that these operations show more flexibility in modeling the long-run and short-run behavior of the time series.

Long-memory property, also called Long Range Dependence (LRD), means the decay or decline of autocorrelation at a polynomial rate or hyperbolic rate, meaning slow decay, because the observations appear to be independent but have non-zero correlations.

The property is statistically clarified assuming a time series (Y_t) that has an autocorrelation function (ρ_k) (Autocorrelations function) and a lag (k) with a sample size (n) and according to the definition of MacLeod and Hippel 1978, the process has the property of long memory if $\lim_{n\to\infty} \sum_{k=-n}^{n} |\rho_k|$ infinite quantity.

It should be noted that the previous is achieved when the integration is a fractional I(d), since (d) is a real number, that is, (0<d<1), and then it is said that the series have a fractional integration, noting that (d) the parameter of the fractional integral to be estimated, which is related to the exponent parameter (Hurst parameter).

As explained, one of the most popular models for modeling the long memory time series are ARFIMA with fractional parameter (d) representing to the difference of the series that are not integer to make it stationary.

The multivariate time series such as VARFIMA model (Vertical Auto Regressive fractional Integration Moving Average model) introduced by Lobato in 1997, with a fractional parameter for each variable can be estimates as a univariate ARFIMA because the fractional parameters in VARFIMA model represented as a diagonal matrix in the arithmetic formula of VARFIMA.

This paper presents different methods of estimation that are Geweke-Porter-Hudak estimator, Smoothed Geweke-Porter-Hudak estimator, Wavelet estimator, Local Whittle estimator and Wavelet Local Whittle estimator.
These methods were compared using mean squared errors (MSE).

2. ARFIMA Model and fractional integration

Given a discrete time series process, Y_t with autocorrelations function γ_i at lag j. According to McLeod and Hippel, the process possesses long memory or is long-range dependent if the sum of the absolute autocorrelations was infinite (decaying to zero slowly at a hyperbolic rate).

$$\lim_{T\to\infty}\sum_{j=-T}^{T}|\rho_j| = \infty \qquad (1)$$

The long memory process has an autocovariance function for large k, given by $\gamma_k \approx \Xi(k)k^{2H-2}$.

The Hurst exponent (H) introduced by Harold Edwin Hurst characterized the long-range dependence (0 < H < 1) and Long-memory occurs when $\frac{1}{2} < H < 1$. (Lildholdt, 2000, Karagiannis and et. al., 2002)

The spectral density function for ARFIMA(p,d,q,) behavior at the origin is found to be:

$$f(\lambda) \sim \frac{\sigma_\varepsilon^2|\psi(1)|^2}{2\pi|\phi(1)|^2}|\lambda|^{-2d} \qquad (2)$$

This may be compared with the leading order behavior of fractional Gaussian noise fGN at the origin given by: (Graves and et. al., 2017, Sheng and et. al., 2010)

$$f(\lambda) \sim c_H|\lambda|^{1-2H} \qquad (3)$$

Then the fractional differencing parameter can therefore be obtained by:

$$d = H - \frac{1}{2} \quad , \quad 0 \le H \le 1 \qquad (4)$$

The closer the value of the Hurst Exponent to 0, the more jagged will the time series be.

The differences or integrated processes that are represented by $I(d)$ are a procedure applied to eliminate nonstationary for the time series and makes it a stationary through finding the differences between the sequential observations ($X_t^* = X_t - X_{t-1}$), then the increment / displacement is called a level difference, and the stationary time series using the differences is called an integrated process.

The process of taking the differences for the time series continue for more than one time until the time series have stationarity. (McCauley and et. al., 2008)

Order of integration (d) is a summary statistic used to describe a unit root process in time series analysis. Specifically, it tells us the minimum number of differences needed to get a stationary series (time series transformed to stationary by differencing d times).

An ideal time series has stationarity. That means that a shift in time does not cause a change in the shape of the distribution. Unit root processes are one cause for nonstationarity. (Kirchgässner and Wolters, 2007)

As mentioned above, the concept of integrated time series should be extended to that effect that the order of integration, d, is no longer restricted to be an integer number. It might be any real number.

The time series is said to be fractionally integrated of order (d), where $(0 < d < 1)$ and transformed into weakly stationary process with strong dependence and slow autocorrelation decay.

In 1980, Granger and Joyeux introduced ARFIMA model that is useful in modeling time series having long memory with fractional differencing parameter $\left(-\frac{1}{2} < d < \frac{1}{2}\right)$ the time series will be covariance stationary, for $\left(0 < d < \frac{1}{2}\right)$ the time series shows a long range depending behavior and for $\left(-\frac{1}{2} < d < 0\right)$ the time series will be Antipersistent. (Dark, 2007)

Granger and Joyeux 1980 have proposed a class of stochastic process by permitting (d) in the ARIMA(p,d,q) process of Box and Jenkins to take any real value. These processes have become very popular due to their ability in providing a good characterization of the long memory properties of many economic and financial time series. The univariate ARFIMA(p,d,q) model represented as: (Kamagaté, and Hili, 2013, Vacha and Barunik, 2012)

$$\phi(L)(1-L)^{d}Y_t = \theta(L)e_t \begin{cases} |d| < 1/2 \\ e_t \text{i.i.d} \sim N(0, \sigma_e^2) \end{cases} \qquad (5)$$

where L: Backshift operator.
$\phi(L)$: AR polynomial of degree (p) with roots outside unit circle.
$\theta(L)$: MA polynomial of degree (q) with roots outside unit circle.
e_t: White noise.

There are multiple extensions of univariate ARFIMA to the multivariate framework. The multivariate generalization would be z_t $(a\ k \times 1)$ vector time series such that: (Sela and Hurvich, 2008)

$$\varphi(L)D(L)Vz_t = \vartheta(L)\varepsilon_t \qquad (6)$$

where $\varphi(L) = \left(\varphi_0 - \varphi_1 L - \varphi_2 L^2 - \cdots - \varphi_p L^p\right)$

$$\vartheta(L) = \left(\vartheta_0 + \vartheta_1 L + \vartheta_2 L^2 + \cdots + \vartheta_q L^q\right)$$

$$\varphi_{i,(L)} = \varphi_i L^i = \begin{bmatrix} \varphi_{i,11} & \varphi_{i,12} & \cdots & \varphi_{i,1k} \\ \varphi_{i,21} & \varphi_{i,22} & \cdots & \varphi_{i,2k} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{i,k1} & \varphi_{k2} & \cdots & \varphi_{i,kk} \end{bmatrix}$$

$$\vartheta_{j,(L)} = \vartheta_i L^i = \begin{bmatrix} \vartheta_{j,11} & \vartheta_{j,12} & \cdots & \vartheta_{j,1k} \\ \vartheta_{i,21} & \vartheta_{i,22} & \cdots & \vartheta_{j,2k} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{j,k1} & \vartheta_{j,k2} & \cdots & \vartheta_{j,kk} \end{bmatrix}$$

$\varphi(L)$ and $\theta(L)$ are $k \times k$ matrix polynomials in the lag operator L.

It will be assumed that $D(L) = diag\left[(1-L)^{d_1}, (1-L)^{d_2}, \ldots, (1-L)^{d_k}\right]$, $\varphi(L)$ is of order (p), $\theta(L)$ is of order (q), $(\varphi(0) = \theta(0) = I_k)$, the roots of $|\varphi(a)|$ and $|\theta(a)|$ are outside the unit circle and $\left(\varepsilon_t \sim IIDN_k(0, \Sigma)\right)$.

The constant $(k \times k)$ matrix (V) is nonsingular. The simple form of the differencing matrix $D(L)$ means that the characteristics of the fractional (z_t) vector series stated below can be obtained by the univariate proofs applied by element. In particular: (z_t) is stationary if $\left(d_i < \frac{1}{2}\right)$ for $(i = 1, 2, \ldots, k)$.

1) (z_t) possess an invertible moving average representation if $d_i > -\frac{1}{2}$.

2) If the spectral density of z_t is denoted $f_z(\lambda)$ then as $(\lambda \to 0)$, $f_z(\lambda) \sim [\kappa_{ij}\lambda^{-(d_i+d_j)}]$ where each (κ_{ij}) is constant and is independent of (d_i) and (d_j).

3) If the autocovariances of (z_t) are denoted $(\gamma_z(s) = E[x_t x'_{t-s}])$ then as $(s \to \infty)$, $\gamma_z(s) \sim [h_{ij} s^{d_i+d_j-1}]$ where each (h_{ij}) is constant and depends on (d_i) and (d_j).

Sowell and Mellon write a general differencing operator as $(1-L)^d$, for $d = \left[-\frac{1}{2}, \frac{1}{2}\right]$ the fractional differencing operator $(1-L)^d$ is defined by its Maclaurin series (binomial theorem) to be:

$$(1-L)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j L^j \tag{7}$$

where $\binom{d}{j}(-1)^j = \frac{\Gamma(d+1)(-1)^j}{\Gamma(d-j+1)\Gamma(j+1)} = \frac{\Gamma(-d+1)}{\Gamma(-d)\Gamma(j+1)}$

Because $\frac{1}{\Gamma(a)}$ is bounded and has roots at the nonpositive integers the sum defining $(1-L)^d$ has finite number of nonzero terms for $d = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $d \neq 0$. (Robinson, 2018)

For a univariate time series, the spectral density measures the contribution of a particular frequency to movements of the time series where for multivariate time series, and the cross-spectral density measures the relationship between two time series at a particular frequency.

3. Estimation methods

As mentioned before, there are many different methods to estimate fractional parameter (d) for ARFIMA model which introduced by Granger and Joyeux (1980) and by Hosking (1981), indirect methods by estimating Hurst exponent (H) which introduced by Mandelbrot and van Ness (1968) then using the relation between H and d as in the formula $\left(H = d + \frac{1}{2}\right)$, such methods Aggregated variance estimator, Differencing variance estimator, Higuchi's method, dretrended fluctuation analysis, Rescaled Range estimator … etc. (Rea and et. al., 2007)

In this paper, some of direct methods estimators of the memory parameter which are semiparametric will be introduced. These methods become popular since they do not require knowing the specific form of the short memory structure. They are based on the periodogram of the series and can be categorized into two types: the log-periodogram (LP) estimator and the local-Whittle (LW) estimator. (Hou and Perron, 2014)

3.1. Geweke-Porter-Hudak estimator

Geweke Porter Hudak method proposed by Geweke and Porter-Hudak (1983) is a semiparametric estimator of (d) based on the first (J) periodogram ordinates for the univariate ARFIMA(p,d,q) as given: (Shang, 2020, Geweke and Hudak, 1983)

$$\hat{d}_{GPH} = \frac{-\frac{1}{2}\sum_{j=1}^{J}\left[\log_{10}(\lambda_j) - \overline{\log_{10}(\lambda_j)}\right]\log_{10} I(\lambda_j)}{\sum_{j=1}^{J}\left[\log_{10}(\lambda_j) - \overline{\log_{10}(\lambda_j)}\right]^2} \quad , \quad j = 1, \dots, J \tag{8}$$

where $\overline{\log_{10}(\lambda_j)} = \frac{1}{J}\sum_{j=1}^{J}\log_{10}(\lambda_j)$

$\lambda_j = \frac{2\pi j}{n}$, $\lambda_j \in [-\pi, \pi]$, set of harmonic frequencies (Fourier frequencies).

$J = \sqrt{n}$, positive integer refers to smallest Fourier frequencies.

$I(\lambda_j) = \frac{1}{2\pi}\{R(0) + 2\sum_{s=1}^{n-1} R(s)\cos(s\lambda_j)\}$, periodogram that is a measure of autocovariance.

$R(s) = \frac{1}{n}\sum_{t=1}^{n-s}(x_t - \bar{x})(x_{t+s} - \bar{x})$, $s = \pm 1, \dots, \pm(n-1)$, sample autocovariance function.

3.2. Smoothed Geweke-Porter-Hudak estimator

Smoothed Geweke-Porter-Hudak estimator introduced by Geweke and Porter-Hudak (1983). They proposed a method for estimating d using a regression model based on the periodogram by using the asymptotic normal distribution of the smoothed periodogram.

A smoothed periodogram was introduced by using Parzen lag window (kernel function) for estimating (d) as: (Reisen, 1994)

$$\hat{d}_{SGPH} = \frac{-\frac{1}{2}\sum_{j=1}^{J}\left[\log_{10}(\lambda_j) - \overline{\log_{10}(\lambda_j)}\right]\log_{10} I_s(\lambda_j)}{\sum_{j=1}^{J}\left[\log_{10}(\lambda_j) - \overline{\log_{10}(\lambda_j)}\right]^2} \quad , \quad j = 1, \dots, J \tag{9}$$

where $I_s(\lambda_j) = \frac{1}{2\pi}\left\{R(0) + 2\sum_{s=1}^{h} K\left(\frac{s}{h}\right) R(s)\cos(s\lambda_j)\right\}$, $\lambda \in [-\pi, \pi]$, smoothed periodogram.

$K(a)$, lag window generator with $-1 < a < 1$, $K(0) = 1$ and $K(-a) = K(a)$.

$$K(a) = \begin{cases} 1 - 6a^2 - 6|a|^3 & |a| \le \frac{1}{2} \\ 2(1 - |a|^3) & -\frac{1}{2} < a \le 1 \\ 0 & |a| > 1 \end{cases} \tag{10}$$

$h = n^{0.9}$, the bandwidth parameter.

Parzen lag window chosen because it always produces positive estimates of the spectral density.

3.3. Wavelet estimator

Wavelet estimator introduced by Tse, Y.K.; Anh, V.V. and Tieng Q., 2002. It is an estimator based on the wavelet theory by applying the discrete wavelet transform (DWT) on time series to obtains the wavelet coefficient $(w_{j,k})$ where they are distributed $N(0, \sigma^2 n_j^{-2d})$, (where this assumption lead to the noncorrelation of wavelet coefficients within the same level as well as across different levels. (Tse, 2002)

Defining the wavelet coefficient's variance at scale (j) as: (Wu, 2020)

$$R(j) = \sigma^2 n_j^{-2d} \quad (11)$$

Then taking the logarithm transformation gotten linear regression model:

$$\log_2 R(j) = \log_2 \sigma^2 + d \log_2 n_j^{-2} \quad (12)$$

For Haar wavelet $n_j = 2^j$ then gotten:

$$\log_2 R(j) = \log_2 \sigma^2 + d(-2j) \quad (13)$$

where $= 1, \dots, J$, No. of the coefficient scale.
$k = 1, \dots, n_j$, time location (No. of wavelet coefficient at transformation level j).
$\log_{10} \sigma^2$ is constant.
$\log_{10} R(j) = \frac{1}{2^j} \sum_{k=1}^{n_j} w_{j,k}^2$, sample variance of wavelet coefficients.

Using ordinary least squares and based on Haar wavelet, (d) can be estimated as given:

$$\hat{d}_w = \frac{J \sum_{j=1}^{J} (-2 \log_2 n_j)(\log_2 R(j)) - \left(\sum_{j=1}^{J} (-2 \log_2 n_j)\right)\left(\sum_{j=1}^{J} \log_2 R(j)\right)}{J \sum_{j=1}^{J} (-2 \log_2 n_j)^2 - \left(\sum_{j=1}^{J} (-2 \log_2 n_j)\right)^2} \quad (14)$$

There are different types of wavelet can used in the estimation; the wavelets with longer filter coefficient can provide a much finer analysis.

The reason of selecting Haar wavelet was the resulting length of coefficients at each DTW level is dyadic. (Wang, 2006)

The estimated parameter $(\hat{d}_w)$ is biased, so the weighted least square is needed and the weight is the reciprocal of the variance of $\log_2 R(j)$ as given: (Wu, 2020, Gong, and et. al., 2000)

$$var(\log_2 R(j)) = \frac{2}{n_j (\ln 2)^2} \quad (15)$$

$$\hat{d}_{wh} = \frac{\sum_{j=1}^{J} h_i \sum_{j=1}^{J} (-2 h_i \log_2 n_j)(\log_2 R(j)) - \left(\sum_{j=1}^{J} -2 h_i \log_2 n_j\right)\left(\sum_{j=1}^{J} h_i \log_2 R(j)\right)}{\left(\sum_{j=1}^{J} h_i\right)\left(\sum_{j=1}^{J} h_i (-2 \log_2 n_j)^2\right) - \left(\sum_{j=1}^{J} -2 h_i \log_2 n_j\right)^2} \quad (16)$$

where $h_i = \frac{1}{var(\log_2 R(j))}$, weights.

3.4. Local Whittle estimator

Local Whittle estimator proposed by Kunsch (1987) and later developed by Robinson (1995a) and Velasco (1999) is a Gaussian semiparametric estimation method based on the approximation periodogram, such as $\left(I_x(\lambda_j) \sim \lambda_j^{-2d} I_u(\lambda_j)\right)$ where (x) represents the time series and (u) represents the error. The estimation of (d) is given as: (Shimotsu and Phillips, 2005)

$$\hat{d}_{lw} = argmin_{d \in \theta} R(d) \quad (17)$$

where $R(d) = \ln\left[\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j)\right] - \frac{2d}{m} \sum_{j=1}^{m} \ln \lambda_j$
$I(\lambda_j)$ is the periodogram of time series.

$\boldsymbol{\theta} = [d_1, d_2]$, closed interval of admissible estimates of fractional parameter, $\left(-\frac{1}{2} < d_1 < d_2 < \frac{1}{2}\right)$, for stationary series $\left(0 < d_1 < d_2 < \frac{1}{2}\right)$.
$\left(m < \frac{n}{2}\right)$ positive integer (number of frequencies used in the minimization) where $\frac{1}{m} + \frac{m}{n} \rightarrow 0 \ as\ n \rightarrow \infty$, (m) less than (n) such as $(m = n^{\alpha}, 0 < \alpha < 1)$ (in this paper taken bandwidth parameter $m = n^{0.8}$ by experimental to be suitable with above conditions for (m)). (Boubaker and Péguin-Feissolle, 2013)

4. Simulation and comparative

In this paper, a data simulated for ARFIMA model used in the estimating of fractional parameter using the methods explained in section 3.

The steps below applied to get results of estimation and make a comparative study between these methods:

1. Applying a MATLAB function [Z]=ARFIMA_model(n,PHi,THi,d,stdx,er) for simulate ARFIMA with different type, in particular consider the cases [ARFIMA(0,d,0), ARFIMA(1,d,0), ARFIMA(0,d,1), ARFIMA(1,d,1)] with $(\varphi_1 = 0.5)$ and $(\theta_1 = 0.5)$, and for different value of fractional integration (0.1, 0.2, 0.3, 0.4) and chosen a sample size to be a dyadic number (2^i) which is suitable when dealing with wavelet estimator (32, 64, 128, 256, 512, 1024).
2. Estimate the fractional parameter for simulated data by introduced methods

➢ Depending on eq.(8) Geweke Porter-Hudak estimator GPH $(\hat{d}_{GPH})$ using MATLAB function dGPHi=dGPH(Z,n).

➢ Depending on eq.(9) Smooted Geweke Porter-Hudak estimator SGPH $(\hat{d}_{SGPH})$ using MATLAB function dSGPHi=dSGPH(Z,n).

➢ Depending on eq.(14) Wavelet estimator $(\hat{d}_w)$ using MATLAB function dwi=dw(Z,n).

➢ Depending on eq.(16) Weighted wavelet estimator $(\hat{d}_{wh})$ using MATLAB function dwhi=dwh(Z,n).

➢ Depending on eq.(17) Local Whittle estimator $(\hat{d}_{lw})$ using MATLAB function dLWi=dLW(Z,n).

3. Step 1 and 2 repeated for (r) iteration, (in this paper r = 500).
4. For each estimation methods, the mean of estimated (d) at each iteration, standard deviation $(\hat{\sigma})$ and mean square error (MSE) computed as given:

$$\hat{d} = \frac{1}{r}\sum_{i=1}^{r} d_i \qquad (18)$$

$$\hat{\sigma} = \sqrt{\frac{1}{r-1}\sum_{i=1}^{r}\left(d_i - \hat{d}\right)^2} \qquad (19)$$

$$MSE = \frac{1}{r}\sum_{i=1}^{r}(d_i - d)^2 \qquad (20)$$

where
$(\hat{d})$ is the mean over all iteration.
(d_i) is the estimated (d) at each iteration.

(d) is the fractional parameter value in the simulated data.

The results of the simulation of fractional parameter estimation (d), and mean square error are shown in below tables.

Table 1: The estimated value $(\hat{d})$ and MSE for n=32, p=0, q=0

ARFIMA(0,0.1,0)			ARFIMA(0,0.2,0)			ARFIMA(0,0.3,0)			ARFIMA(0,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1275	0.1684	GPH	0.1875	0.1894	GPH	0.2701	0.1715	GPH	0.4241	0.184
SGPH	0.0096	0.0906	SGPH	0.0791	0.1027	SGPH	0.1649	0.1063	SGPH	0.2779	0.1092
Wavelet	-0.127	0.1794	Wavelet	-0.042	0.1795	Wavelet	0.0181	0.1988	Wavelet	0.1571	0.1537
Wwavelet	-0.059	0.0749	Wwavelet	0.0114	0.0867	Wwavelet	0.0844	0.0949	Wwavelet	0.1962	0.0854
Local W.	0.1614	0.0116	Local W.	0.1934	0.0105	Local W.	0.2473	0.016	Local W.	0.3006	0.0216

Table 2: The estimated value $(\hat{d})$ and MSE for n=64, p=0, q=0

ARFIMA(0,0.1,0)			ARFIMA(0,0.2,0)			ARFIMA(0,0.3,0)			ARFIMA(0,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1035	0.1204	GPH	0.1929	0.1165	GPH	0.3133	0.1198	GPH	0.4095	0.1207
SGPH	0.0333	0.0607	SGPH	0.1095	0.0625	SGPH	0.2026	0.0758	SGPH	0.3079	0.0707
Wavelet	-0.083	0.0885	Wavelet	0.0156	0.0948	Wavelet	0.1092	0.093	Wavelet	0.2215	0.0768
Wwavelet	-0.017	0.03	Wwavelet	0.0655	0.0362	Wwavelet	0.1488	0.0395	Wwavelet	0.2453	0.0395
Local W.	0.1369	0.005	Local W.	0.1873	0.0077	Local W.	0.2561	0.0108	Local W.	0.3309	0.0115

Table 3: The estimated value $(\hat{d})$ and MSE for n=128, p=0, q=0

ARFIMA(0,0.1,0)			ARFIMA(0,0.2,0)			ARFIMA(0,0.3,0)			ARFIMA(0,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.0958	0.0767	GPH	0.1787	0.0759	GPH	0.3143	0.0665	GPH	0.4233	0.0729
SGPH	0.0369	0.0476	SGPH	0.1091	0.048	SGPH	0.2316	0.0428	SGPH	0.3344	0.0457
Wavelet	-0.051	0.0543	Wavelet	0.0331	0.0605	Wavelet	0.1499	0.0554	Wavelet	0.2394	0.0604
Wwavelet	0.0197	0.0132	Wwavelet	0.101	0.0174	Wwavelet	0.1943	0.0188	Wwavelet	0.2795	0.0216
Local W.	0.1255	0.0024	Local W.	0.1859	0.0056	Local W.	0.2774	0.0061	Local W.	0.3534	0.0054

Table 4: The estimated value $(\hat{d})$ and MSE for n=256, p=0, q=0

ARFIMA(0,0.1,0)			ARFIMA(0,0.2,0)			ARFIMA(0,0.3,0)			ARFIMA(0,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1001	0.0466	GPH	0.2207	0.0424	GPH	0.2997	0.0489	GPH	0.4391	0.0433
SGPH	0.0533	0.0275	SGPH	0.1602	0.0244	SGPH	0.2448	0.0328	SGPH	0.3736	0.0292
Wavelet	-0.019	0.039	Wavelet	0.085	0.0347	Wavelet	0.172	0.0395	Wavelet	0.2822	0.0352
Wwavelet	0.042	0.0067	Wwavelet	0.1313	0.0077	Wwavelet	0.2143	0.0107	Wwavelet	0.3071	0.0121
Local W.	0.1197	0.0014	Local W.	0.1913	0.003	Local W.	0.2821	0.0039	Local W.	0.3675	0.0031

Table 5: The estimated value $(\hat{d})$ and MSE for n=512, p=0, q=0

ARFIMA(0,0.1,0)			ARFIMA(0,0.2,0)			ARFIMA(0,0.3,0)			ARFIMA(0,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1291	0.0293	GPH	0.2189	0.025	GPH	0.3317	0.0302	GPH	0.4317	0.0275
SGPH	0.086	0.0168	SGPH	0.1761	0.0164	SGPH	0.2895	0.0176	SGPH	0.3943	0.0187
Wavelet	0.0062	0.0242	Wavelet	0.1052	0.0222	Wavelet	0.2079	0.0238	Wavelet	0.2999	0.0243
Wwavelet	0.0564	0.0031	Wwavelet	0.1426	0.0048	Wwavelet	0.2323	0.006	Wwavelet	0.3209	0.0077
Local W.	0.1143	0.0007	Local W.	0.1913	0.002	Local W.	0.2931	0.002	Local W.	0.3779	0.0013

Table 6: The estimated value $(\hat{d})$ and MSE for n=1024, p=0, q=0

ARFIMA(0,0.1,0)			ARFIMA(0,0.2,0)			ARFIMA(0,0.3,0)			ARFIMA(0,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1044	0.0207	GPH	0.2075	0.0193	GPH	0.3147	0.0182	GPH	0.4248	0.0183
SGPH	0.0811	0.0123	SGPH	0.1785	0.0126	SGPH	0.2847	0.012	SGPH	0.3968	0.012
Wavelet	0.0211	0.0155	Wavelet	0.1083	0.0183	Wavelet	0.2047	0.0174	Wavelet	0.2924	0.0225
Wwavelet	0.0681	0.0016	Wwavelet	0.1533	0.0028	Wwavelet	0.2396	0.0043	Wwavelet	0.3334	0.0051
Local W.	0.112	0.0004	Local W.	0.1966	0.0011	Local W.	0.2958	0.0011	Local W.	0.3861	0.0006

Table 7: The estimated value $(\hat{d})$ and MSE for n=32, p=0, q=1

ARFIMA(0,0.1,1)			ARFIMA(0,0.2,1)			ARFIMA(0,0.3,1)			ARFIMA(0,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.2089	0.1932	GPH	0.2696	0.1825	GPH	0.369	0.1929	GPH	0.5004	0.2375
SGPH	0.0949	0.0826	SGPH	0.142	0.0897	SGPH	0.2344	0.0912	SGPH	0.338	0.1026
Wavelet	0.0679	0.1155	Wavelet	0.1086	0.1237	Wavelet	0.2088	0.1255	Wavelet	0.2973	0.1255
Wwavelet	0.2068	0.0565	Wwavelet	0.2496	0.0479	Wwavelet	0.332	0.0456	Wwavelet	0.418	0.0496
Local W.	0.3634	0.0747	Local W.	0.3792	0.0356	Local W.	0.389	0.0099	Local W.	0.395	0.0008

Table 8: The estimated value $(\hat{d})$ and MSE for n=64, p=0, q=1

ARFIMA(0,0.1,1)			ARFIMA(0,0.2,1)			ARFIMA(0,0.3,1)			ARFIMA(0,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.0981	0.1201	GPH	0.2264	0.1228	GPH	0.3247	0.1128	GPH	0.4379	0.1117
SGPH	0.0265	0.0651	SGPH	0.1323	0.0649	SGPH	0.2187	0.0668	SGPH	0.3232	0.0676
Wavelet	0.0511	0.0668	Wavelet	0.1484	0.056	Wavelet	0.2219	0.0664	Wavelet	0.3128	0.0641
Wwavelet	0.2145	0.0316	Wwavelet	0.2956	0.0256	Wwavelet	0.364	0.0211	Wwavelet	0.4407	0.0183
Local W.	0.3367	0.0631	Local W.	0.3754	0.0335	Local W.	0.3917	0.0093	Local W.	0.3984	0.0000

Table 9: The estimated value $(\hat{d})$ and MSE for n=128, p=0, q=1

ARFIMA(0,0.1,1)			ARFIMA(0,0.2,1)			ARFIMA(0,0.3,1)			ARFIMA(0,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1123	0.0713	GPH	0.2143	0.079	GPH	0.3152	0.0835	GPH	0.4151	0.0665
SGPH	0.0484	0.0428	SGPH	0.1337	0.0458	SGPH	0.2448	0.0465	SGPH	0.3287	0.0444
Wavelet	0.0621	0.0338	Wavelet	0.1435	0.044	Wavelet	0.2412	0.0363	Wavelet	0.3147	0.0453
Wwavelet	0.2305	0.0234	Wwavelet	0.3	0.0173	Wwavelet	0.3851	0.0138	Wwavelet	0.4584	0.0112
Local W.	0.2839	0.0397	Local W.	0.3555	0.0275	Local W.	0.3902	0.0089	Local W.	0.3977	0.0002

Table 10: The estimated value $(\hat{d})$ and MSE for n=256, p=0, q=1

ARFIMA(0,0.1,1)			ARFIMA(0,0.2,1)			ARFIMA(0,0.3,1)			ARFIMA(0,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1181	0.0429	GPH	0.2222	0.0446	GPH	0.3312	0.0434	GPH	0.4365	0.0459
SGPH	0.0632	0.0265	SGPH	0.1673	0.0292	SGPH	0.2705	0.0248	SGPH	0.3772	0.0292
Wavelet	0.08	0.0203	Wavelet	0.1718	0.0196	Wavelet	0.2588	0.0229	Wavelet	0.3513	0.0271
Wwavelet	0.2425	0.0234	Wwavelet	0.3224	0.018	Wwavelet	0.405	0.0141	Wwavelet	0.4904	0.0115
Local W.	0.2264	0.0198	Local W.	0.321	0.0177	Local W.	0.3844	0.0079	Local W.	0.3989	0.0001

Table 11: The estimated value $(\hat{d})$ and MSE for n=512, p=0, q=1

ARFIMA(0,0.1,1)			ARFIMA(0,0.2,1)			ARFIMA(0,0.3,1)			ARFIMA(0,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1049	0.0262	GPH	0.209	0.0255	GPH	0.3228	0.0307	GPH	0.4164	0.0285
SGPH	0.0634	0.0187	SGPH	0.1765	0.016	SGPH	0.2817	0.0199	SGPH	0.3864	0.0185
Wavelet	0.0759	0.015	Wavelet	0.1717	0.0138	Wavelet	0.2621	0.015	Wavelet	0.3523	0.0177
Wwavelet	0.2485	0.0233	Wwavelet	0.3317	0.0187	Wwavelet	0.4103	0.0135	Wwavelet	0.4972	0.011
Local W.	0.1926	0.0104	Local W.	0.2921	0.0107	Local W.	0.3758	0.0066	Local W.	0.3994	0.0000

Table 12: The estimated value $(\hat{d})$ and MSE for n=1024, p=0, q=1

ARFIMA(0,0.1,1)			ARFIMA(0,0.2,1)			ARFIMA(0,0.3,1)			ARFIMA(0,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1068	0.018	GPH	0.2175	0.017	GPH	0.3204	0.0182	GPH	0.4354	0.0183
SGPH	0.083	0.0123	SGPH	0.1883	0.0102	SGPH	0.294	0.0127	SGPH	0.4031	0.0121
Wavelet	0.0739	0.0102	Wavelet	0.1701	0.0095	Wavelet	0.2561	0.0111	Wavelet	0.347	0.0117
Wwavelet	0.2574	0.0254	Wwavelet	0.337	0.0195	Wwavelet	0.418	0.0146	Wwavelet	0.5031	0.0113
Local W.	0.1676	0.0056	Local W.	0.2673	0.0058	Local W.	0.3622	0.0047	Local W.	0.3998	0.000

Table 13: The estimated value $(\hat{d})$ and MSE for n=32, p=1, q=0

ARFIMA(1,0.1,0)			ARFIMA(1,0.2,0)			ARFIMA(1,0.3,0)			ARFIMA(1,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.0322	0.1768	GPH	0.108	0.1785	GPH	0.2355	0.1652	GPH	0.3346	0.1863
SGPH	-0.074	0.1009	SGPH	0.0049	0.1161	SGPH	0.0907	0.1179	SGPH	0.1807	0.1343
Wavelet	-0.35	0.3299	Wavelet	-0.251	0.3163	Wavelet	-0.19	0.3666	Wavelet	-0.079	0.3376
Wwavelet	-0.386	0.2891	Wwavelet	-0.308	0.307	Wwavelet	-0.245	0.3476	Wwavelet	-0.149	0.3555
Local W.	0.1	0.0000	Local W.	0.1025	0.0097	Local W.	0.1072	0.0383	Local W.	0.1159	0.0831

Table 14: The estimated value $(\hat{d})$ and MSE for n=64, p=1, q=0

ARFIMA(1,0.1,0)			ARFIMA(1,0.2,0)			ARFIMA(1,0.3,0)			ARFIMA(1,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.0508	0.1149	GPH	0.1503	0.1514	GPH	0.2929	0.1374	GPH	0.3558	0.1277
SGPH	-0.016	0.0765	SGPH	0.065	0.0801	SGPH	0.1855	0.0777	SGPH	0.2498	0.0841
Wavelet	-0.276	0.2101	Wavelet	-0.175	0.203	Wavelet	-0.067	0.1998	Wavelet	0.0179	0.2044
Wwavelet	-0.321	0.1951	Wwavelet	-0.235	0.2082	Wwavelet	-0.143	0.214	Wwavelet	-0.066	0.2341
Local W.	0.1	0	Local W.	0.1014	0.0098	Local W.	0.1088	0.0373	Local W.	0.1279	0.0768

Table 15: The estimated value $(\hat{d})$ and MSE for n=128, p=1, q=0

ARFIMA(1,0.1,0)			ARFIMA(1,0.2,0)			ARFIMA(1,0.3,0)			ARFIMA(1,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1028	0.0787	GPH	0.1983	0.064	GPH	0.2932	0.0718	GPH	0.3908	0.0781
SGPH	0.0358	0.0447	SGPH	0.114	0.0431	SGPH	0.2198	0.0484	SGPH	0.3116	0.0531
Wavelet	-0.18	0.1104	Wavelet	-0.09	0.1187	Wavelet	0.0026	0.1229	Wavelet	0.0961	0.1276
Wwavelet	-0.257	0.1343	Wwavelet	-0.176	0.1491	Wwavelet	-0.084	0.1551	Wwavelet	-0.005	0.1714
Local W.	0.1001	0.0000	Local W.	0.1028	0.0096	Local W.	0.1246	0.0324	Local W.	0.1813	0.0528

Table 16: The estimated value $(\hat{d})$ and MSE for n=256 , p=1 ,q=0

ARFIMA(1,0.1,0)			ARFIMA(1,0.2,0)			ARFIMA(1,0.3,0)			ARFIMA(1,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.0953	0.0476	GPH	0.1905	0.0452	GPH	0.3227	0.0508	GPH	0.41	0.0474
SGPH	0.045	0.0299	SGPH	0.1398	0.0303	SGPH	0.2616	0.0337	SGPH	0.348	0.0342
Wavelet	-0.131	0.0741	Wavelet	-0.036	0.0777	Wavelet	0.0687	0.0774	Wavelet	0.1602	0.0799
Wwavelet	-0.221	0.1063	Wwavelet	-0.136	0.1164	Wwavelet	-0.054	0.1289	Wwavelet	0.0359	0.136
Local W.	0.1002	0.0000	Local W.	0.1077	0.0089	Local W.	0.1628	0.0217	Local W.	0.2485	0.027

Table 17: The estimated value $(\hat{d})$ and MSE for n=512, p=1, q=0

ARFIMA(0,0.1,1)			ARFIMA(0,0.2,1)			ARFIMA(0,0.3,1)			ARFIMA(0,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.107	0.0259	GPH	0.2147	0.0273	GPH	0.3207	0.0254	GPH	0.4367	0.0295
SGPH	0.0657	0.0168	SGPH	0.1704	0.0184	SGPH	0.2808	0.0178	SGPH	0.393	0.0189
Wavelet	-0.088	0.0492	Wavelet	0.0061	0.0515	Wavelet	0.1069	0.0521	Wavelet	0.2197	0.0458
Wwavelet	-0.198	0.0897	Wwavelet	-0.114	0.0999	Wwavelet	-0.025	0.1072	Wwavelet	0.0683	0.1116
Local W.	0.1	0.0000	Local W.	0.1178	0.0074	Local W.	0.2014	0.0119	Local W.	0.3022	0.0115

Table 18: The estimated value $(\hat{d})$ and MSE for n=1024, p=1, q=0

ARFIMA(1,0.1,0)			ARFIMA(1,0.2,0)			ARFIMA(1,0.3,0)			ARFIMA(1,0.4,0)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.102	0.0181	GPH	0.216	0.0196	GPH	0.3265	0.0202	GPH	0.43	0.017
SGPH	0.0756	0.0119	SGPH	0.19	0.0121	SGPH	0.2962	0.0117	SGPH	0.4003	0.0108
Wavelet	-0.07	0.04	Wavelet	0.0314	0.039	Wavelet	0.1328	0.0376	Wavelet	0.225	0.0397
Wwavelet	-0.183	0.0809	Wwavelet	-0.097	0.0888	Wwavelet	-0.009	0.0962	Wwavelet	0.0839	0.1006
Local W.	0.1001	0.0000	Local W.	0.1326	0.0053	Local W.	0.2297	0.0062	Local W.	0.3281	0.0064

Table 19: The estimated value $(\hat{d})$ and MSE for n=32, p=1, q=1

ARFIMA(1,0.1,1)			ARFIMA(1,0.2,1)			ARFIMA(1,0.3,1)			ARFIMA(1,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.0841	0.1963	GPH	0.2205	0.1954	GPH	0.2837	0.1936	GPH	0.4102	0.1846
SGPH	-0.008	0.098	SGPH	0.0906	0.0961	SGPH	0.1672	0.1101	SGPH	0.2689	0.1056
Wavelet	-0.116	0.1669	Wavelet	-0.066	0.212	Wavelet	0.0281	0.1855	Wavelet	0.1473	0.1673
Wwavelet	-0.059	0.073	Wwavelet	-0.003	0.0958	Wwavelet	0.08	0.0954	Wwavelet	0.1833	0.0924
Local W.	0.1529	0.0098	Local W.	0.1926	0.0103	Local W.	0.2441	0.0161	Local W.	0.296	0.0227

Table 20: The estimated value $(\hat{d})$ and MSE for n=64, p=1, q=1

ARFIMA(1,0.1,1)			ARFIMA(1,0.2,1)			ARFIMA(1,0.3,1)			ARFIMA(1,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.0856	0.1182	GPH	0.1766	0.1254	GPH	0.3011	0.1137	GPH	0.3945	0.1177
SGPH	0.0126	0.0647	SGPH	0.0854	0.0736	SGPH	0.2043	0.0719	SGPH	0.2889	0.0789
Wavelet	-0.072	0.086	Wavelet	0.0064	0.0945	Wavelet	0.0915	0.1082	Wavelet	0.1775	0.1068
Wwavelet	-0.011	0.0297	Wwavelet	0.065	0.0353	Wwavelet	0.1455	0.0421	Wwavelet	0.2257	0.0484
Local W.	0.1361	0.005	Local W.	0.1904	0.0077	Local W.	0.2583	0.0114	Local W.	0.3202	0.0142

Table 21: The estimated value $(\hat{d})$ and MSE for n=128, p=1, q=1

ARFIMA(1,0.1,1)			ARFIMA(1,0.2,1)			ARFIMA(1,0.3,1)			ARFIMA(1,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1073	0.0734	GPH	0.2027	0.0735	GPH	0.3188	0.0699	GPH	0.4183	0.0683
SGPH	0.0441	0.0408	SGPH	0.1229	0.0477	SGPH	0.2274	0.0453	SGPH	0.3285	0.048
Wavelet	-0.046	0.0567	Wavelet	0.0561	0.0542	Wavelet	0.1436	0.0585	Wavelet	0.2428	0.0566
Wwavelet	0.0221	0.013	Wwavelet	0.1048	0.0163	Wwavelet	0.1899	0.0197	Wwavelet	0.2749	0.0226
Local W.	0.1271	0.0027	Local W.	0.1842	0.005	Local W.	0.275	0.0071	Local W.	0.3496	0.006

Table 22: The estimated value $(\hat{d})$ and MSE for n=256, p=1, q=1

ARFIMA(1,0.1,1)			ARFIMA(1,0.2,1)			ARFIMA(1,0.3,1)			ARFIMA(1,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1138	0.0459	GPH	0.2078	0.0437	GPH	0.3197	0.047	GPH	0.4241	0.0436
SGPH	0.0598	0.028	SGPH	0.1543	0.0301	SGPH	0.2557	0.0299	SGPH	0.3683	0.03
Wavelet	-0.005	0.0323	Wavelet	0.0916	0.0301	Wavelet	0.1823	0.0351	Wavelet	0.2779	0.0394
Wwavelet	0.0451	0.0058	Wwavelet	0.1316	0.0075	Wwavelet	0.216	0.01	Wwavelet	0.3044	0.0125
Local W.	0.1212	0.0016	Local W.	0.1911	0.0033	Local W.	0.2847	0.0038	Local W.	0.3684	0.0027

Table 23: The estimated value $(\hat{d})$ and MSE for n=512, p=1, q=1

ARFIMA(1,0.1,1)			ARFIMA(1,0.2,1)			ARFIMA(1,0.3,1)			ARFIMA(1,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.116	0.0292	GPH	0.2234	0.0235	GPH	0.3104	0.0303	GPH	0.4278	0.0307
SGPH	0.0822	0.0183	SGPH	0.1785	0.0168	SGPH	0.271	0.0203	SGPH	0.3948	0.019
Wavelet	0.0063	0.0233	Wavelet	0.1051	0.0242	Wavelet	0.1947	0.0261	Wavelet	0.2943	0.0276
Wwavelet	0.0591	0.003	Wwavelet	0.1441	0.0044	Wwavelet	0.2305	0.0064	Wwavelet	0.3215	0.0078
Local W.	0.1168	0.0008	Local W.	0.1967	0.0018	Local W.	0.2895	0.0022	Local W.	0.3782	0.0014

Table 24: The estimated value $(\hat{d})$ and MSE for n=1024, p=1, q=1

ARFIMA(1,0.1,1)			ARFIMA(1,0.2,1)			ARFIMA(1,0.3,1)			ARFIMA(1,0.4,1)		
Method	Dhat	MSE	Method	Dhat	MSE	Method	dhat	MSE	Method	dhat	MSE
GPH	0.1143	0.0191	GPH	0.2102	0.0167	GPH	0.3235	0.0175	GPH	0.4169	0.019
SGPH	0.0829	0.0115	SGPH	0.1788	0.0111	SGPH	0.2897	0.0103	SGPH	0.3905	0.0128
Wavelet	0.0136	0.0192	Wavelet	0.1091	0.0184	Wavelet	0.2035	0.019	Wavelet	0.2985	0.0189
Wwavelet	0.0695	0.0016	Wwavelet	0.1538	0.0027	Wwavelet	0.2409	0.0041	Wwavelet	0.3324	0.0052
Local W.	0.1144	0.0006	Local W.	0.1967	0.0011	Local W.	0.2961	0.0011	Local W.	0.3846	0.0007

5. Discussion of Results

In this paper, a time series is generated through an ARFIMA(p,d,q) model as in eq.(5), and different cases are covered when fractional parameter (d= 0.1, 0.2, 0.3, 0.4) and when sample size (n= 32, 64, 128, 256, 512, 1024) and establish the short-memory parameters (constants in the definition) $(\varphi_1 = 0.5)$ and $(\theta_1 = 0.5)$ that are smaller than one, so the simulated time series is stationary and invertible.

The mean square error (MSE) computed for each estimation's method at different cases where the (MSE) provides some information on the accuracy of estimated long memory parameter.

Through the Analyze of simulation results:

1. By comparing from tables (1) to table (6) when (p=0, q=0) and for all sample size (32, 64, 128, 256, 512, 1024) with different values of (d= 0.1, 0.2, 0.3 0.4) found that the best estimation method is Local Whittle and the smallest mean square error (MSE = 0.0004) for the model ARFIMA(0,0.1,0) with sample size (n = 1024).

2. By comparing from tables (7) to table (12) when (p=0, q=1) and for all sample size (32, 64, 128, 256, 512, 1024) with different values of (d= 0.1, 0.2, 0.3 0.4) found that the best estimation method is Weighted Wavelet for tables (7, 8, 9) when (d = 0.1, 0.2) and Local Whittle when (d = 0.3, 0.4), while tables (10, 11, 12) the best estimation method is Local Whittle and the smallest mean square error (MSE = 0.0000) for the models ARFIMA(0,0.4,1) with (n = 64), ARFIMA(0,0.4,1) with (n = 512), ARFIMA(0,0.4,1) with (n = 1024).

3. By comparing from tables (13) to table (18) when (p=1, q=0) and for all sample size (32, 64, 128, 256, 512, 1024) with different values of (d= 0.1, 0.2, 0.3 0.4) found that the best estimation method is Local Whittle and the smallest mean square error (MSE = 0) for the model ARFIMA(1,0.1,0) with sample size (n =32, 64,128,256,512,1024).

4. By comparing from tables (19) to table (24) when (p=1, q=1) and for all sample size (32, 64, 128, 256, 512, 1024) with different values of (d= 0.1, 0.2, 0.3 0.4) found that the best estimation method is Local Whittle and the smallest mean square error (MSE = 0.0006) for the model ARFIMA(1,0.1,1) with sample size (n = 1024).

In general, from the tables (1) to (24), the Local Whittle method has the smallest (MSE) except for a table (7), n=32 ARFIM (0,0.1,1), table (8), n=64 ARFIM (0,0.1,1), ARFIM (0,0.2,1) and table (9) when n=128, ARFIM (0,0.1,1) and ARFIM (0,0.2,1) the Weighted wavelet has the smallest (MSE).

So, from results and under the assumed variables in the simulation it can consider that Local Whittle method is the best method for estimating fractional parameter of ARFIMA model.

Different value of short-memory can affect the accuracy of estimated fractional parameters, so in table 7, 8, 9 the value of short-memory parameter affect the best estimation method that was weighted wavelet for this table.

Sometimes chosen an inappropriate orthogonal wavelet type caused unstable in the best method for simulated models of ARFIMA which caused by correlations of wavelet coefficients, so this paper used Haar wavelet that is the simplest type and when using a higher type of wavelet (another type such as Daubechies, Symlets, …) may be get that Wavelet or Weighted Wavelet methods are better than Local Whittle method this can be as a future work.

In this paper, many variables needed to assume in the simulation, so as a future work it can make the simulation of ARFIMA with different value of (φ_1) and (θ_1)or simulate a higher degree of ARFIMA model and note its effect.

6. Conclusions

The value of the fractional differences parameter ranges between (-0.5<d<0.5) and all the estimated values of (d) in all tables within this range, and the non-convergence of the estimated values of (d) with the values imposed in the simulation to each table is due to the fact that all the estimation methods for (d) are approximate methods,.

The difference in the estimation between methods can be attributed to the accuracy of the method.

It is noted in most tables that the best method (Local W.) gave reasonable results for the estimated (d) values compared to the imposed (d) values and the estimators are within the specified range of (d).

Depending on the simulation results for all sample sizes the following conclusions were reached:

1. Noted that the mean squared error decreases as the sample size increases and for all methods.

2. In general, and for almost tables, the Local Whittle is the best methods for all sizes and the all value of (p) and (q).

3. In general, as shown in the result, many researchers stated that there is no specific and better method to estimate fractional parameter (d) where the used method depends on the type and nature of data or time series.

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دراسة مقارنة لبعض طرائق تقدير معلمة التكامل الكسري في نموذج أرفيما

الباحث/ عمار مؤيد صابر
كلية الإدارة والاقتصاد/جامعة بغداد
ammar.m@coadec.uobaghdad.edu.iq

أ.م.د. رباب عبد الرضا صالح
كلية الإدارة والاقتصاد/جامعة بغداد
rabab.saleh@coadec.uobaghdad.edu.iq

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مستخلص البحث :

يعد تحليل الذاكرة الطويلة أحد أكثر المجالات نشاطًا في الإقتصاد القياسي والسلاسل الزمنية حيث تم تقديم طرق مختلفة لتحديد وتقدير معلمة الذاكرة الطويلة في سلاسل زمنية متكاملة كسريا.

أحد أكثر النماذج شيوعًا المستخدمة لتمثيل السلاسل الزمنية التي لها ذاكرة طويلة هي نماذج ARFIMA (أنموذج الإنحدار الذاتي والوسط المتحرك المتكامل كسريا) حيث تتمثل هذه الذاكرة برقم كسري يسمى معلمة التكامل الكسرية.

لتحليل وتحديد أنموذج ARFIMA ، يجب تقدير المعلمة الكسرية. هناك العديد من الطرائق لتقدير المعاملات الكسرية. في هذا البحث تم تقسيم طرائق التقدير إلى طرائق غير مباشرة حيث يتم تقدير معامل هورست (H) أولاً ثم يتم تقدير معامل التكامل الكسري (d) من خلال العلاقة بينهما. بالنسبة للطرائق المباشرة ، يتم تقدير معامل التكامل الكسري بشكل مباشر دون الإعتماد على معلمة هورست، ومعظمها طرائق شبه معلمية.

في هذا البحث تم إستخدام بعض الطرائق المباشرة الأكثر شيوعًا لتقدير معلمة اتكامل الكسري وهي (Geweke-Porter-Hudak و Geweke-Porter-Hudak الموزونة و Local Whittle والمويجية والمويجة الموزونة) بإستخدام طريقة المحاكاة لقيم مختلفة من (d) وحجوم مختلفة من السلاسل الزمنية.

تمت المقارنة بين الطرائق باستخدام متوسط الخطأ التربيعي (MSE). إتضح أن أفضل الطرائق لتقدير معلمة التكامل الكسري هي (Local Whittle).

تم المحاكاة لأنموذج ARFIMA والطرائق المستخدمة في البحث بواسطة دوال تم برمجتها بواسطة برنامج ماتلاب R2020a.

المصطلحات الرئيسة للبحث: السلاسل الزمنية ، أس هيرست ، أنموذج ARFIMA ، الفروق ، التكامل الكسري ، التحويل المويجي ، الذاكرة طويلة المدى.

*البحث مستل من رسالة ماجستير