Under Different Priors &Two Loss Functions To Compare Bayes Estimators With Some of Classical Estimators For the Parameter of Exponential Distribution

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Abstract

In this study, different estimators were used for estimating parameter of the exponential distribution, such as maximum likelihood estimator, moment estimator and the Bayes estimator, by assuming six types when the prior distribution for the scale parameter is: Levy distribution, Gumbel type-II distribution, Inverse Chi-square distribution, Inverted Gamma distribution, improper distribution, Non-informative distribution .Under squared and weighted squared error loss functions. We used simulation technique, to compare the performance for each estimator, several cases from Exponential distribution for data generating, for different samples sizes (small, medium, and large). Simulation results shown that The best method is the bayes estimation according to the smallest values of MSE & MWSE for all samples sizes (n) comparative to the estimated values by using Maximum likelihood estimation method (MLE) and Moment estimation method (ME). According to obtained results, we see that when the prior distribution for θ is Inverted Gamma distribution for some values of the parameters $\alpha \& \beta$, given the best results according to the smallest values of MSE & MWSE comparative to the same values which obtained by using MLE& ME for the assuming true values by $\theta = 0.5$ and for all samples sizes. When the prior distribution for θ is Improper distribution for some values of the parameters a & b, given the best results according to the smallest values of MSE & MWSE comparative to the same values which obtained by using MLE & ME for the assuming true values by $\theta = 1,2,4$ and all samples sizes.

Key words: The Exponential, Maximum likelihood method, Moment estimation method, Bayes method, mean squared errors (MSE), mean weighted squared errors (MWSE).

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1. Introduction

The difference between Maximum Likelihood estimation and Bayesian estimation is that in maximum likelihood estimation the parameters are not random variables. In Bayesian analysis the unknown parameter is regarded as being the value of a random variable from a given probability distribution, with the knowledge of some information about its value prior to observing the data x_1, x_2… x_n (Ross, 2009) [9] ; we mention some of studies in a brief manner:

In (1998) Rossman, Short, and Parks [10] presented some thought provoking insights on the relationship between Bayesian and classical estimation using the continuous uniform distribution.

In (2001) Elfessi and Reineke [7] intended to explore these relationships using the exponential distribution. They show how the classical estimators can be obtained from various choices made within a Bayesian framework.

In (2005) Ali and Woo and Nadarajah [5] considered bayes estimators of the parameter of the standard exponential distribution. They derived bayes estimators under a symmetric squared error loss function as well as an asymmetric loss function.

In (2005) Al_Kutubi [2] studied the extension of Jeffery prior information with square error loss function in exponential distribution.

In (2007) Abu-Taleb and Smadi and Alawneh [1] considered exponential survival time with the exponential random censor time. They derive bayes estimates assuming the inverted gamma prior along with the bayesian credible intervals.

In (2009) Al_Kutubi and Ibrahim [3] provided the extension of Jeffery prior information with new loss function for estimating the parameter of exponential distribution of life time .Through simulation study the performance of their estimator was compared to the standard bayes with Jeffery Prior information with respect to the mean square error (MSE) and mean percentage error (MPE).

In (2009) Al_Kutubi and Ibrahim [4] annexed Jeffery prior information to get the modify bayes estimator and then compared it with standard Bayes estimator and maximum likelihood estimator to find the best (less MSE and MPE). They derived Bayesian and Maximum likelihood of the scale parameter and survival functions. Simulation study was used to compare between estimators and Mean Square Error (MSE) and Mean Percentage Error (MPE) of estimators are computed.

In (2010) Tahir and Aslam [11] provided the comparison of uninformative (Jeffrey's and uniform) priors for the parameter of the exponential model for time-to-failure data. They also presented Bayesian and classical analysis of the model. Their comparison is based upon the posterior variance, the bayesian point and interval estimates, the coefficients of skewness of the posterior distribution and the posterior predictive distribution.

In (2013) Yang and Zhou and Yuan [15] studied the bayes estimation of parameter of exponential distribution under a bounded loss function, named reflected gamma loss function, which proposed by Towhidi and Behboodian (1999). They used the inverse Gamma prior distribution as the prior distribution of the parameter of exponential distribution. Bayesian estimators are obtained under squared error loss and the reflected gamma loss functions.

The aim of research

The objective of this research, we try to find best method to estimate parameter of exponential distribution .According to the smallest value of Mean Square Errors (MSE) and Mean Weighted Square Errors (MWSE) were calculated to compare the methods of estimation. We used the maximum likelihood estimator, the moment estimator and the bayes estimator by assuming six types of priors, to get bayes estimation: Levy distribution, Gumbel type-II distribution, Inverse Chi-square distribution, Inverted Gamma distribution, Improper distribution, and Non-informative distribution when the Bayesian estimation based on squared and weighted squared error loss functions.

Several cases from exponential distribution for data generating ,of different samples sizes (small, medium, and large).The results were obtained by using simulation technique, Programs written using MATLAB-R2008a program were used.

2.1 Exponential Distribution

Let us consider $x_1, x_2, \ldots, x_n$ is a random sample of n independent observations from an exponential distribution having the probability density function (pdf) define as [7,8]:

$$f(x;\theta) = \theta^{-1} \exp(-\frac{x}{\theta}) \quad , \quad x > 0 \qquad \ldots (1)$$

where $\theta > 0$ is mean, standard deviation, and scale parameter of the distribution, θ is a survival parameter in the sense that if a random variable x is the duration of time that a given biological or mechanical system manages to survive and $x \sim Exp(\theta)$ then $E[x] = \theta$. That is to say, the expected duration of survival of the system is θ units of time.

2.2 Parameter Estimation Methods

In this section, we used several methods to estimation parameter θ.

2.3.1 Maximum likelihood Estimation(MLE)

From the Exponential pdf given in (1) the likelihood function will be as follows[6]:

$$L(\underline{x} \backslash \theta) = \prod_{i=1}^{n} f(x;\theta) = \theta^{-n} \exp(-\frac{\sum_{i=1}^{n} x_i}{\theta}) \qquad \ldots (2)$$

By taking the log and differentiating partially with respect to θ, we get:

$$\frac{\partial}{\partial\theta} \log L(\underline{x} \backslash \theta) = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} \qquad \ldots (3)$$

Then the MLE of θ is the solution of equation (2) after equating the first derivative to zero, Hence:

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} \qquad \ldots (4)$$

2.3.2. Moments Estimation (ME)

The method of moments is another technique commonly used in the field of estimation of parameters. If $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a random sample of size (n) represent a set of data, then an unbiased estimator for the r^th^ origin moment is [6]:

$$m_r = \frac{\sum_{i=1}^{n} x_i^r}{n} \qquad \ldots(5)$$

Where m_r stands for the r^{th} sample moment. The first moment of the Exponential distribution as:

$$M_1 = E(x) = \frac{1}{(1/\theta)} = \theta \qquad \ldots(6)$$

Therefore by equating sample and population moments we get

$$m_1 = M_1 = E(x) = \frac{1}{(1/\theta)} = \theta \qquad \ldots(7)$$

From (7) we get $\bar{x} = \theta \quad \Rightarrow \quad \hat{\theta}_{MM} = \bar{x} \qquad \ldots(8)$

2.3.3 Bayes Estimation Method

Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a random sample of size n with probability density function given in equation (1) and likelihood function given in equation (2).In this paper , we derived the posterior distributions for the unknown parameter θ using the following six types of priors ,and then get bayes estimation [8]:

1. **Levy distribution.**
2. **Gumbel type-II distribution [12].**
3. **Inverse Chi-square distribution [14].**
4. **Inverted Gamma distribution [13].**
5. **Improper distribution.**
6. **Non-informative distribution.**

2.3.3.1 The posterior distribution using different Priors

In this section, we derive the posterior distributions .It is assumed that θ follows six types of prior distributions with pdf as given in table below:

Table -1: The six types of prior distributions ($P(\theta)$) with pdf for θ .

Prior distribution	$P(\theta)$
θ **~Levy(** b_3 **)**	$P(\theta)\ \alpha\ \sqrt{\frac{b_3}{2\pi}}\ \theta^{-\frac{3}{2}}\ \exp(-\frac{b_3}{2\theta})$ for $b_3,\ \theta > 0$
θ **~Gumbel type-II(** b **)**	$P(\theta)\ \alpha\ b\ \theta^{-2}\ \exp(-\frac{b}{\theta})$ for $b,\ \theta > 0$
θ **~Inverse Chi-square(** v **)**	$P(\theta)\ \alpha\ \frac{1}{2^{\frac{v}{2}}}\ \theta^{-\frac{v}{2}-1}\ \exp(-\frac{1}{2\theta})$ for $v,\ \theta > 0$
θ **~Inverted Gamma(** α, β **)**	$P(\theta)\ \alpha\ \frac{\beta^{\alpha}}{\Gamma\alpha}\ \theta^{-(\alpha+1)}\ \exp(-\frac{\beta}{\theta})$ for $\alpha, \beta,\ \theta > 0$
θ **~Improper(** a, b **)**	$P(\theta)\ \alpha\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta})$ for $b,\ \theta > 0$ and $-\infty < a < \infty$
θ **~Non-informative(** c **)**	$P(\theta)\ \alpha\ \frac{1}{\theta^{c}}$ for $\theta, c > 0$

Then the posterior distribution of given the data $\underline{x} = (x_1, x_2, \ldots, x_n)$ **is[7]:**

$$P(\theta \setminus x) = \frac{L(\underline{x} \setminus \theta)\ \ P(\theta)}{\int_{\theta} L(\underline{x} \setminus \theta)\ \ P(\theta) d\theta} \qquad \ldots(9)$$

Substituting the equation (2) and for each $P(\theta)$ **as shown in table -1 in equation (9), we get the posterior distributions for the unknown parameter θ are derived using the following six types of priors (for more details see Appendix-A).**

Table -2: The posterior distributions ($P(\theta \backslash x)$) for the unknown parameter (θ) are derived using the following six types of priors.

Prior dist^{n}.	The posterior distribution ($P(\theta \backslash x)$)
Levy	$P_1(\theta \backslash x) \sim$ Inverted Gamma $(\alpha_{(new)} = (n+\frac{1}{2}), \beta_{(new)} = (\sum_{i=1}^{n} x_i + \frac{b_3}{2}))$ with pdf $P_1(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + \frac{b_3}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})} \theta^{-[(n+\frac{1}{2})+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{b_3}{2}))$ $n,\ b_3,\ \theta > 0$
Gumbel type-II	$P_2(\theta \backslash x) \sim$ Inverted Gamma $(\alpha_{(new)} = (n+1), \beta_{(new)} = (\sum_{i=1}^{n} x_i + b))$ with pdf $P_2(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + b)^{(n+1)} \theta^{-[(n+1)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + b))}{\Gamma(n+1)}$ $n,\ b,\ \theta > 0$
Inverse Chi-square	$P_3(\theta \backslash x)) \sim$ Inverted Gamma $(\alpha_{(new)} = (n+\frac{v}{2}), \beta_{(new)} = (\sum_{i=1}^{n} x_i + \frac{1}{2}))$ with pdf $P_3(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + \frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \frac{1}{2}))$ $n,\ v,\ \theta > 0$
Inverted Gamma	$P_4(\theta \backslash x) \sim$ Inverted Gamma $(\alpha_{(new)} = (n+\alpha), \beta_{(new)} = (\sum_{i=1}^{n} x_i + \beta))$ with pdf $P_4(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + \beta)^{(n+\alpha)} \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + \beta))}{\Gamma(n+\alpha)}$ $n, \beta, \alpha, \theta > 0$
Improper	$P_5(\theta \backslash x) \sim$ Inverted Gamma $(\alpha_{(new)} = (n+a), \beta_{(new)} = (\sum_{i=1}^{n} x_i + b))$ with pdf $P_5(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i + b)^{(n+a)} \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i + b))}{\Gamma(n+a)}$ $n,\ b,\ \theta > 0$ and $-\infty < a < \infty$

Non-informative	$P_6(\theta \backslash x) \sim$ Inverted Gamma $(\alpha_{(new)} = (n+c-1), \beta_{(new)} = (\sum_{i=1}^{n} x_i))$ with pdf $$P_6(\theta \backslash x) = \frac{(\sum_{i=1}^{n} x_i)^{(n+c-1)} \theta^{-[(n+c-1)+1]} \exp(-\frac{1}{\theta}\sum_{i=1}^{n} x_i)}{\Gamma(n+c-1)}$$ $n, c \ , \ \theta > 0$

2.3.3.2 <u>Bayes' Estimators</u>

In this section, we derive Bayes' estimators for the scale parameter θ, it was considered with six different priors and under two loss functions:

1. **The squared error loss function** $L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$.
2. **The weighted squared error loss function** $L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta}$.

Where $\hat{\theta}$ is an estimator for θ, was considered with different six priors, and under two loss functions. Following is the derivation of these estimators:

<u>First: The squared error loss function</u>

In this section, we derive Bayes' estimator. To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$$L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \qquad \dots (10)$$

After simplified steps, we get Bayes estimator of θ denoted by $\hat{\theta}_{SE}$ for the above prior as follows

$$\hat{\theta}_{SE} = E(\theta \backslash x) = \int_0^{\infty} \theta \, P(\theta \backslash x) \, d\theta \qquad \dots (11)$$

So, the following results are the derivations of these estimators under the squared error loss function with different six priors (for more details see Appendix-B).

Table -3: The estimators ($\hat{\theta}_{SE}$) under the squared error loss function with different six priors.

Prior distribution	$\hat{\theta}_{SE} = E(\theta \backslash x) = \int_0^\infty \theta\, P(\theta \backslash x)\, d\theta$
Levy	$\hat{\theta}_{SE1} = \frac{\Gamma(n - \frac{1}{2})}{\Gamma(n + \frac{1}{2})} (\sum_{i=1}^{n} x_i + \frac{b_3}{2})$, $n \,\&\, b_3 > 0$
Gumbel type-II	$\hat{\theta}_{SE2} = \frac{\Gamma(n)}{\Gamma(n+1)} (\sum_{i=1}^{n} x_i + b)$, $n \,\&\, b > 0$
Inverse Chi-square	$\hat{\theta}_{SE3} = \frac{\Gamma(n + \frac{v}{2} - 1)}{\Gamma(n + \frac{v}{2})} (\sum_{i=1}^{n} x_i + \frac{1}{2})$, $n \,\&\, v > 0$
Inverted Gamma	$\hat{\theta}_{SE4} = \frac{\Gamma(n + \alpha - 1)}{\Gamma(n + \alpha)} (\sum_{i=1}^{n} x_i + \beta)$, $n, \beta, \alpha > 0$
Improper	$\hat{\theta}_{SE5} = \frac{\Gamma(n + a - 1)}{\Gamma(n + a)} (\sum_{i=1}^{n} x_i + b)$, $n, b, a > 0$
Non-informative	$\hat{\theta}_{SE6} = \frac{\Gamma(n + c - 2)}{\Gamma(n + c - 1)} (\sum_{i=1}^{n} x_i)$, $n, c > 0$

Second: The weighted squared error loss function

In this section, we derive Bayes' estimator .To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$$L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta} \qquad \dots (12)$$

After simplified steps, we get Bayes estimator of θ denoted by $\hat{\theta}_{WSE}$ for the above prior as follows

$$\hat{\theta}_{WSE} = \frac{1}{E(\frac{1}{\theta} \backslash x)} = \frac{1}{\int_0^\infty \frac{1}{\theta} P(\theta \backslash x)\, d\theta} \qquad \dots (13)$$

So, the following results are the derivations of these estimators under the weighted squared error loss function with different six priors (for more details see Appendix-C).

Table -4: The estimators ($\hat{\theta}_{WSE}$) under the weighted squared error loss function with different six priors.

Prior distribution	$\hat{\theta}_{WSE} = \dfrac{1}{E(\frac{1}{\theta} \backslash x)} = \dfrac{1}{\int_0^\infty \frac{1}{\theta} P(\theta \backslash x)\, d\theta}$
Levy	$\hat{\theta}_{WSE1} = \dfrac{\Gamma(n+\frac{1}{2})(\sum_{i=1}^{n} x_i + \frac{b_3}{2})}{\Gamma(n+\frac{3}{2})} \quad , n \,\&\, b_3 > 0$
Gumbel type-II	$\hat{\theta}_{WSE2} = \dfrac{\Gamma(n+1)(\sum_{i=1}^{n} x_i + b)}{\Gamma(n+2)} \quad , n \,\&\, b > 0$
Inverse Chi-square	$\hat{\theta}_{WSE3} = \dfrac{\Gamma(n+\frac{v}{2})(\sum_{i=1}^{n} x_i + \frac{1}{2})}{\Gamma(n+\frac{v}{2}+1)} \quad n, v > 0$
Inverted Gamma	$\hat{\theta}_{WSE4} = \dfrac{\Gamma(n+\alpha)(\sum_{i=1}^{n} x_i + \beta)}{\Gamma(n+\alpha+1)} \quad n, \beta, \alpha > 0$
Improper	$\hat{\theta}_{WSE5} = \dfrac{\Gamma(n+a)(\sum_{i=1}^{n} x_i + b)}{\Gamma(n+a+1)} \quad n, b, a > 0$
Non-informative	$\hat{\theta}_{WSE6} = \dfrac{\Gamma(n+c-1)(\sum_{i=1}^{n} x_i)}{\Gamma(n+c)} \quad n, c > 0$

3. Simulation Study

In this study, we have generated random samples from Exponential distribution and compared the performance of MLE and MME and Bayes estimator based on them. So we have considered several steps to perform simulation study as follow:

1. **We have chosen sample size n = 10, 25, 50 and 100 to represent small, moderate and large sample size.**
2. **We generated data from exponential distribution for the scale parameter; we have considered randomly several values for the parameter of exponential distribution** $\theta = 0.5, 1, 2, 4$.
3. **We used randomly three values for the parameter of the Levy distribution (b_3=0.5, 1, 2) as prior distribution for** θ.
4. **We used randomly three values for the parameter of the Gumbel type-II distribution (b=2, 3,5) as prior distribution for** θ.

5. **We used randomly three values for the parameter of the Inverse Chi-square distribution (v=2, 4, 6) as prior distribution for** θ **.**

6. **We used randomly two values for the parameters of the Inverted Gamma distribution (** $\alpha = 2, 3$ **&** $\beta = 0.5, 1$**) as prior distribution for** θ **.**

7. **We used randomly three values for the parameters of the Improper distribution (a=1, 2, 3 & b=1, 2, 3) as prior distribution for** θ **.**

8. **We used randomly three values for the function of the non-informative prior distribution c = 1, 2, 3.**

9. **The number of replication used was (** $r = 1000$ **) for each sample size (n).**

10. **We obtained estimators for scale parameter from equations (4), (8) and also the estimators in table -3, it means the estimators** $(\hat{\theta}_{SE})$ **under the squared error loss function with six different priors .And the estimators in table -4, it means the estimators** $(\hat{\theta}_{WSE})$ **under the weighted squared error loss function with different six priors.**

The simulation program was written by using MATLAB-R2008a program. After the parameter θ **was estimated, Mean Square Errors (MSE) and Mean weighted squared Errors (MWSE) were calculated to compare the methods of estimation, where:**

- $MSE = \frac{1}{r}\sum_{r=1}^{1000} (\hat{\theta}(r) - \theta)^2$ ...(14)
- $MWSE = \frac{1}{r}\sum_{r=1}^{1000} [(\hat{\theta}(r) - \theta)^2 / \theta]$ ...(15)

See appendix-D, for the programs algorithm. The results of the simulation study are summarized and tabulated in tables (4.1-4.4).In each row of tables (4.1-4.4) ,we have four estimated values for θ $(\hat{\theta})$ **with MSE for all samples sizes (n) and values (** b_3 **, b, v,** α, β**, a , b, c) respectively. Also the results of the simulation study are summarized and tabulated in tables (4.5-4.8).In each row of tables (4.5-4.8) ,we have four estimated values for** θ $(\hat{\theta})$ **with MWSE for all samples sizes (n) and values (**b_3**,b,v,** α, β **, a, b, c,) respectively. By using different estimation methods that is maximum likelihood estimator and the moment estimator .And the Bayes estimators in six types of prior distribution .So our criteria is the best method that gives the smallest value of (MSE) and (MWSE). We list the results in the following tables (4.1 -4.8).**

In general, as we see in the tables (4.1-4.8) by using different estimation methods, we find the Mean Square Errors (MSE) and Mean weighted squared Errors (MWSE) were decreased when sample size increased in all cases .And we obtained the same results for MSE& MWSE by using maximum likelihood estimation (MLE) and the moment estimation(ME) for all sample sizes (n), because they have the same formula see formula from equations (4), (8).

we see in the tables (4.1-4.4) that we obtained un appropriate estimated values for θ $(\hat{\theta})$, when the prior distribution for θ is levy distribution, for all assuming values for b_3& for the true values for $\theta = 0.5, 1, 2, 4$, and for all samples sizes (n) comparative to the estimated values by using MLE and ME, according to the smallest values of MSE for all samples sizes (n).

So we obtained over estimated values for θ $(\hat{\theta})$, when the prior distribution for θ is Gumbel type-II distribution, for all assuming values for b and $\theta = 0.5, 1, 2, 4$, and for all samples sizes (n) comparative to the estimated values by using MLE and ME, according to the smallest values of MSE for all samples sizes (n).

<u>In table (4.1),</u> when the true value of $\theta(\theta = 0.5)$ in general , we obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the estimated values by using MLE and ME .we listed them when the prior distribution for θ are

- **Inverse Chi-square distribution with v=6.**
- **Inverted Gamma distribution with ($\alpha = 3, \beta = 1$).**
- **Improper distribution with (a=3, b=1).**
- **Non-informative distribution with c=3.**

<u>In table (4.2),</u> when the true value of $\theta(\theta = 1)$ in general , we obtained a good estimation according to the smallest values of MSE for all samples sizes (n) ,comparative to the estimated values by using MLE and ME .we listed them when the prior distribution for θ are

- **Inverse Chi-square distribution with v=4&6.**
- **Inverted Gamma distribution with ($\alpha = 3, \beta = 1$).**
- **Improper distribution with (a=3, b=2).**
- **Non-informative distribution with c=3.**

<u>In tables (4.3) & (4.4),</u> when the true value of $\theta(\theta = 2)$ & $(\theta = 4)$ in general , we obtained a good estimation according to the smallest values of MSE for all samples sizes (n), comparative to the estimated values by using MLE and ME .we listed them when the prior distribution for θ are

- **Inverse Chi-square distribution with v=4.**
- **Inverted Gamma distribution with ($\alpha = 2, \beta = 1$).**
- **Improper distribution with (a=b=3).**
- **Non-informative distribution with c=3.See tables (4.1-4.4) .**

Table 4.1: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

Method	parameters			Estimate for θ ($\hat{\theta}$) Sample Size(n) 10	25	50	100	MSE Sample Size(n) 10	25	50	100
	θ	-	-	10	25	50	100	10	25	50	100
MLE	0.5	-	-	0.4918	0.5030	0.5017	0.5001	0.0225	0.0103	0.0052	0.0026
ME	0.5	-	-	0.4918	0.5030	0.5017	0.5001	0.0225	0.0103	0.0052	0.0026
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta \backslash x)$)							
	0.5	0.5	-	5.1683	12.826	25.333	50.258	24.035	158.35	629.56	2502.2
		1	-	5.4183	13.076	25.583	50.508	26.431	164.57	642.04	2527.1
		2	-	5.9183	13.576	26.083	51.008	31.6	177.4	667.37	2577.4
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta \backslash x)$)							
	0.5	2	-	0.6918	0.5830	0.5417	0.5201	0.0592	0.0177	0.0069	0.0030
		3	-	0.7918	0.6230	0.5617	0.5301	0.1076	0.0254	0.0089	0.0035
		5	-	0.9918	0.7030	0.6017	0.5501	0.2643	0.0515	0.0155	0.0051
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta \backslash x)$)							
	0.5	2	-	0.54183	0.5230	0.5117	0.5051	0.0242	0.0108	0.0053	0.0027
		4	-	0.49257	0.5029	0.5016	0.5001	0.0186	0.0095	0.0049	0.0026
		6	-	0.45152	0.7030	0.4919	0.4952	0.0179	0.0515	0.0048	0.0025
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta \backslash x)$)							
	0.5	2	0.5	0.4926	0.5029	0.5016	0.5001	0.0186	0.0095	0.0049	0.0026
		2	1	0.5380	0.5222	0.5114	0.5050	0.019	0.0099	0.0051	0.0026
		3	0.5	0.4515	0.4843	0.4919	0.4952	0.0179	0.0091	0.0048	0.0026
		3	1	0.4932	0.5028	0.5016	0.5001	0.0156	0.0088	0.0047	0.0025
Bayes	θ	a	b	Improper distribution ($P_5(\theta \backslash x)$)							
	0.5	1	1	0.4918	0.5430	0.5217	0.5101	0.0225	0.0121	0.0056	0.0027
		1	2	0.4471	0.5830	0.5417	0.5201	0.0213	0.0172	0.0069	0.0030
		1	3	0.7918	0.6230	0.5617	0.5301	0.1076	0.0254	0.0089	0.0035
		2	1	0.5380	0.5222	0.5114	0.5050	0.0199	0.0099	0.0047	0.0026
		2	2	0.6289	0.5606	0.5114	0.5149	0.0352	0.0132	0.0051	0.0028
		2	3	0.7198	0.5991	0.5506	0.5248	0.0669	0.0193	0.0075	0.0032
		3	1	0.4932	0.5028	0.5016	0.5001	0.0156	0.0088	0.0048	0.0025
		3	2	0.5765	0.5399	0.5208	0.5001	0.0214	0.0104	0.0052	0.0025
		3	3	0.6599	0.5769	0.5401	0.5001	0.0411	0.0147	0.0064	0.0025
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta \backslash x)$)							
	0.5	1	-	0.5465	0.524	0.5119	0.5051	0.0298	0.0117	0.0055	0.0027
		2	-	0.4918	0.5030	0.5017	0.5001	0.0225	0.0103	0.0052	0.0026
		3	-	0.4471	0.4837	0.4918	0.4951	0.0213	0.0098	0.0050	0.0026

Table 4.2: Shows the values for square error loss function (MSE).

Method	parameters			Estimate for θ ($\hat{\theta}$) Sample Size(n) 10	25	50	100	MSE Sample Size(n) 10	25	50	100
	θ	-	-	10	25	50	100	10	25	50	100
MLE	1	-	-	1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
ME	1	-	-	1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta \backslash x)$)							
	1	0.5	-	10.323	25.435	50.003	100.1	97.112	622.08	2453.1	9920.5
		1	-	10.573	25.685	50.253	100.35	101.84	634.36	2477.7	9970.1
		2	-	11.073	26.185	50.753	100.85	111.66	659.29	2527.2	10070
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta \backslash x)$)							
	1	2	-	1.2073	1.0874	1.0351	1.0185	0.1448	0.0476	0.0219	0.0104
		3	-	1.3073	1.1274	1.0551	1.0285	0.1963	0.0562	0.0237	0.0108
		5	-	1.5073	1.2074	1.0951	1.0485	0.3592	0.0830	0.0297	0.0124
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta \backslash x)$)							
	1	2	-	1.0573	1.0274	1.0051	1.0035	0.1051	0.0407	0.0207	0.0100
		4	-	0.9612	0.9878	0.9853	0.9935	0.0857	0.0371	0.0201	0.0099
		6	-	0.8811	0.9512	0.9664	0.9837	0.0848	0.0366	0.0202	0.0099
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta \backslash x)$)							
	1	2	0.5	0.9612	0.9878	0.9853	0.9935	0.0857	0.0371	0.0201	0.0099
		2	1	1.0067	1.0071	0.9951	0.9984	0.0842	0.0370	0.0199	0.0098
		3	0.5	0.8811	0.9512	0.9664	0.9837	0.0848	0.0366	0.0202	0.0099
		3	1	0.9227	0.9698	0.9760	0.9886	0.0767	0.0352	0.0197	0.0097
Bayes	θ	a	b	Improper distribution ($P_5(\theta \backslash x)$)							
	1	1	1	1.1073	1.0474	1.0151	1.0085	0.1134	0.0422	0.0209	0.0101
		1	2	1.2073	1.0874	1.0351	1.0185	0.1448	0.0476	0.0219	0.0104
		1	3	1.3073	1.1274	1.0551	1.0285	0.1963	0.0562	0.0237	0.0108
		2	1	1.0067	1.0071	0.9951	0.9984	0.0842	0.0370	0.0199	0.0098
		2	2	1.0976	1.0456	1.0148	1.0084	0.0937	0.0390	0.0201	0.0099
		2	3	1.1885	1.084	1.0344	1.0183	0.1197	0.0440	0.0211	0.0101
		3	1	0.9227	0.9698	0.9760	0.9886	0.0767	0.0352	0.0197	0.0097
		3	2	1.0061	1.0068	0.9952	0.9984	0.0707	0.0343	0.0191	0.0096
		3	3	1.0894	1.0439	1.0145	1.0083	0.0787	0.0362	0.0193	0.0097
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta \backslash x)$)							
	1	1	-	1.1192	1.0494	1.0154	1.0085	0.1400	0.0458	0.0218	0.0103
		2	-	1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
		3	-	0.9157	0.9686	0.9755	0.9885	0.0913	0.0379	0.0205	0.0099

Table4.3: Shows the values for $\hat{\theta}$ under square square error loss function (MSE).

Method	parameters			Estimate for θ ($\hat{\theta}$)				MSE			
				Sample Size(n)				Sample Size(n)			
	θ	-	-	10	25	50	100	10	25	50	100
MLE	2	-	-	2.0036	2.0018	2.0041	1.9954	0.39642	0.1607	0.0781	0.0409
ME	2	-	-	2.0036	2.0018	2.0041	1.9954	0.39642	0.1607	0.0781	0.0409
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta\backslash x)$)							
	2	0.5	-	20.286	50.296	100.46	199.79	374.02	2432.9	9888.7	39529
		1	-	20.536	50.546	100.71	200.04	383.22	2457.1	9938	39628
		2	-	21.036	51.046	101.21	200.54	402.01	2505.9	10037	39827
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta\backslash x)$)							
	2	2	-	2.2036	2.0818	2.0441	2.0154	0.4378	0.1674	0.0800	0.0411
		3	-	2.3036	2.1218	2.0641	2.0254	0.4885	0.1755	0.0822	0.0415
		5	-	2.5036	2.2018	2.1041	2.0454	0.6500	0.2014	0.0889	0.0429
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta\backslash x)$)							
	2	2	-	2.0536	2.0218	2.0141	2.0004	0.3992	0.1611	0.0783	0.0408
		4	-	1.8669	1.9441	1.9746	1.9806	0.3453	0.1517	0.0757	0.0404
		6	-	1.7113	1.8721	1.9366	1.9612	0.3586	0.1541	0.0762	0.0407
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta\backslash x)$)							
	2	2	0.5	1.8669	1.9441	1.9746	1.9806	0.3453	0.1517	0.0757	0.0404
		2	1	1.9124	1.9633	1.9844	1.9855	0.3352	0.1499	0.0753	0.0402
		3	0.5	1.7113	1.8721	1.9366	1.9612	0.3586	0.1541	0.0762	0.0407
		3	1	1.753	1.8906	1.9463	1.9661	0.3362	0.1497	0.0751	0.0404
Bayes	θ	a	b	Improper distribution ($P_5(\theta\backslash x)$)							
	2	1	1	2.1036	2.0418	2.0241	2.0054	0.4071	0.1624	0.0786	0.0409
		1	2	2.2036	2.0818	2.0441	2.0154	0.4378	0.1674	0.0800	0.0411
		1	3	2.3036	2.1218	2.0641	2.0254	0.4885	0.1755	0.0822	0.0415
		2	1	1.9124	1.9633	1.9844	1.9855	0.3352	0.1499	0.0753	0.0402
		2	2	2.0033	2.0018	2.004	1.9954	0.3276	0.1485	0.0750	0.0400
		2	3	2.0942	2.0402	2.0236	2.0053	0.3364	0.1502	0.0756	0.0401
		3	1	1.753	1.8906	1.9463	1.9661	0.3362	0.1497	0.0751	0.0404
		3	2	1.8363	1.9276	1.9655	1.9759	0.3020	0.1430	0.0734	0.0398
		3	3	1.9197	1.9647	1.9847	1.9857	0.2817	0.1390	0.0724	0.0394
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta\backslash x)$)							
	2	1	-	2.2262	2.0852	2.045	2.0155	0.5405	0.1816	0.0833	0.0419
		2	-	2.0036	2.0018	2.0041	1.9954	0.3964	0.1607	0.0781	0.0409
		3	-	1.8215	1.9248	1.9648	1.9756	0.3594	0.1542	0.0763	0.0406

Table 4.4: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

Method	parameters			Estimate for θ ($\hat{\theta}$)				MSE			
				Sample Size(n)				Sample Size(n)			
	θ	-	-	10	25	50	100	10	25	50	100
MLE	4	-	-	4.0072	4.0037	4.0133	3.998	1.5857	0.6428	0.3301	0.1581
ME	4	-	-	4.0072	4.0037	4.0133	3.998	1.5857	0.6428	0.3301	0.1581
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta\backslash x)$)							
	4	0.5	-	40.322	100.34	200.91	400.05	1477.9	9683.4	39600	$1.584e^{+5}$
		1	-	40.572	100.59	201.16	400.3	1496.1	9731.6	39698	$1.586e^{+5}$
		2	-	41.072	101.09	201.66	400.8	1532.9	9828.5	39896	$1.590e^{+5}$
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta\backslash x)$)							
	4	2	-	4.2072	4.0837	4.0533	4.018	1.6286	0.6498	0.3328	0.1584
		3	-	4.3072	4.1237	4.0733	4.028	1.68	0.6581	0.3353	0.1589
		5	-	4.5072	4.2037	4.1133	4.048	1.8429	0.6842	0.3428	0.1604
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta\backslash x)$)							
	4	2	-	4.0572	4.0237	4.0233	4.003	1.5889	0.6433	0.3305	0.1581
		4	-	3.6884	3.8689	3.9444	3.9634	1.4076	0.6115	0.3203	0.1563
		6	-	3.381	3.7256	3.8685	3.9245	1.4843	0.6264	0.3224	0.1576
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta\backslash x)$)							
	4	2	0.5	3.6884	3.8689	3.9444	3.9634	1.4076	0.6115	0.3203	0.1563
		2	1	3.7338	3.8881	3.9542	3.9683	1.3813	0.6068	0.3193	0.1560
		3	0.5	3.381	3.7256	3.8685	3.9245	1.4843	0.6264	0.3224	0.1576
		3	1	3.4227	3.7441	3.8781	3.9294	1.4345	0.6165	0.3199	0.1569
Bayes	θ	a	b	Improper distribution ($P_5(\theta\backslash x)$)							
	4	1	1	4.1072	4.0437	4.0333	4.008	1.5971	0.6447	0.3311	0.1582
		1	2	4.2072	4.0837	4.0533	4.018	1.6286	0.6498	0.3328	0.1584
		1	3	4.3072	4.1237	4.0733	4.028	1.68	0.6581	0.3353	0.1589
		2	1	3.7338	3.8881	3.9542	3.9683	1.3813	0.6068	0.3193	0.1560
		2	2	3.8247	3.9266	3.9738	3.9782	1.3412	0.5997	0.3178	0.1554
		2	3	3.9156	3.9651	3.9934	3.9881	1.3176	0.5955	0.3172	0.1551
		3	1	3.4227	3.7441	3.8781	3.9294	1.4345	0.6165	0.3199	0.1569
		3	2	3.506	3.7812	3.8974	3.9392	1.3452	0.599	0.3156	0.1556
		3	3	3.5893	3.8182	3.9166	3.949	1.2698	0.5841	0.3120	0.1545
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta\backslash x)$)							
	4	1	-	4.4524	4.1705	4.0952	4.0384	2.1623	0.7265	0.3526	0.1628
		2	-	4.0072	4.0037	4.0133	3.998	1.5857	0.6428	0.3301	0.1581
		3	-	3.6429	3.8497	3.9346	3.9584	1.438	0.6169	0.3214	0.1567

<u>In table (4.5),</u> we obtained over estimated values for θ $(\hat{\theta})$, when the prior distribution for θ is Gumbel type-II distribution, for all assuming values for b, when the true value of $\theta(\theta=0.5)$, and for all samples sizes (n) comparative to the estimated values by using MLE and ME according to the smallest values of MWSE for all samples sizes (n).

But in general , we obtained a good estimation according to the smallest values of MWSE for all samples sizes (n) comparative to the estimated values by using MLE and ME .we listed them when the prior distribution for θ are

- **Levy distribution with b_3=0.5.**
- **Inverse Chi-square distribution with v=4.**
- **Inverted Gamma distribution with ($\alpha=3,\beta=1$).**
- **Improper distribution with (a=3, b=1).**
- **Non-informative distribution with c=2.**

<u>In table (4.6)</u>, when the true value of $\theta(\theta=1)$ in general , we obtained a good estimation according to the smallest values of MWSE for all samples sizes (n) comparative to the estimated values by using MLE and ME .we listed them when the prior distribution for θ are

- **Levy distribution with b_3=1.**
- **Gumbel type-II distribution with b=2.**
- **Inverse Chi-square distribution with v=4.**
- **Inverted Gamma distribution with ($\alpha=2,\beta=1$).**
- **Improper distribution with (a=b=3).**
- **Non-informative distribution with c=2.**

<u>In table (4.7),</u> when the true value of $\theta(\theta=2)$ in general , we obtained a good estimation according to the smallest values of MWSE for all samples sizes (n) ,comparative to the estimated values by using MLE and ME .we listed them when the prior distribution for θ are

- **Levy distribution with b_3=2.**
- **Gumbel type-II distribution with b=2.**
- **Inverse Chi-square distribution with v=2.**
- **Inverted Gamma distribution with ($\alpha=2,\beta=1$).**
- **Improper distribution with (a=2, b=3).**
- **Non-informative distribution with c=2.**

<u>In table (4.8),</u> when the true value of $\theta\,(\theta = 4)$ in general , we obtained a good estimation according to the smallest values of MWSE for all samples sizes (n) ,comparative to the estimated values by using MLE and ME .we listed them when the prior distribution for θ are

- **Levy distribution with b_3=2.**
- **Gumbel type-II distribution with b=3.**
- **Inverse Chi-square distribution with v=2.**
- **Inverted Gamma distribution with ($\alpha = 2, \beta = 1$).**
- **Improper distribution with (a=1, b=3).**
- **Non-informative distribution with c=2.**

Table 4.5: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

Method	parameters			Estimate for θ ($\hat{\theta}$)				MWSE			
				Sample Size(n)				Sample Size(n)			
	θ	-	-	10	25	50	100	10	25	50	100
MLE	0.5	-	-	0.4918	0.5030	0.5017	0.5001	0.04497	0.0206	0.0103	0.0053
ME	0.5	-	-	0.4918	0.5030	0.5017	0.5001	0.04497	0.0206	0.0103	0.0053
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta \setminus x)$)							
	0.5	0.5	-	0.4922	0.5029	0.5016	0.5001	0.0407	0.0197	0.0101	0.0052
		1	-	0.5160	0.5127	0.5065	0.5025	0.0411	0.0201	0.0101	0.0052
		2	-	0.5636	0.5323	0.5164	0.5075	0.0487	0.0218	0.0106	0.0053
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta \setminus x)$)							
	0.5	2	-	0.6289	0.5606	0.5310	0.5149	0.0703	0.0263	0.0118	0.0056
		3	-	0.7198	0.5991	0.5506	0.5248	0.1337	0.0386	0.0150	0.0063
		5	-	0.9016	0.676	0.5898	0.5446	0.3597	0.0809	0.0260	0.0091
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta \setminus x)$)							
	0.5	2	-	0.4926	0.5029	0.5016	0.5001	0.0372	0.0190	0.0099	0.0051
		4	-	0.4515	0.4843	0.4919	0.4951	0.0358	0.0181	0.0096	0.0051
		6	-	0.4167	0.467	0.4827	0.4903	0.0403	0.0185	0.0097	0.0051
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta \setminus x)$)							
	0.5	2	0.5	0.4515	0.4843	0.4919	0.4951	0.0358	0.0181	0.0096	0.0051
		2	1	0.4932	0.5028	0.5015	0.5001	0.0312	0.0176	0.0095	0.0050
		3	0.5	0.4167	0.467	0.4827	0.4903	0.0403	0.0185	0.0097	0.0051
		3	1	0.4552	0.4848	0.4921	0.4952	0.0305	0.0168	0.0093	0.0050
Bayes	θ	a	b	Improper distribution ($P_5(\theta \setminus x)$)							
	0.5	1	1	0.5380	0.5221	0.5114	0.5050	0.0399	0.0199	0.0101	0.0052
		1	2	0.6289	0.5606	0.5310	0.5149	0.0703	0.0263	0.0118	0.0056
		1	3	0.7198	0.5990	0.5506	0.0051	0.1337	0.0386	0.0150	0.0063
		2	1	0.4931	0.5028	0.5015	0.5001	0.0312	0.0176	0.0095	0.0051
		2	2	0.5765	0.5398	0.5208	0.5098	0.0428	0.0207	0.0104	0.0052
		2	3	0.6598	0.5768	0.5400	0.5196	0.0822	0.0294	0.0127	0.0058
		3	1	0.4552	0.4848	0.4921	0.4952	0.0305	0.0168	0.0093	0.0050
		3	2	0.5321	0.5205	0.511	0.5049	0.0286	0.0172	0.0094	0.0050
		3	3	0.6091	0.5562	0.5298	0.5146	0.0503	0.0227	0.0109	0.0053
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta \setminus x)$)							
	0.5	1	-	0.4918	0.5030	0.5016	0.5000	0.0449	0.0205	0.0103	0.0052
		2	-	0.4471	0.4836	0.4918	0.4951	0.0426	0.0195	0.0100	0.0052
		3	-	0.4098	0.4657	0.4823	0.4902	0.0473	0.0199	0.0101	0.0052

Table 4.6: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

Method	parameters			Estimate for θ ($\hat{\theta}$)				MWSE			
				Sample Size(n)				Sample Size(n)			
	θ	-	-	10	25	50	100	10	25	50	100
MLE	1	-	-	1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
ME	1	-	-	1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta \setminus x)$)							
	1	0.5	-	0.9831	0.9974	0.9901	0.9959	0.0927	0.0384	0.0204	0.0099
		1	-	1.007	1.0073	0.9951	0.9984	0.0924	0.0385	0.0203	0.0099
		2	-	1.0546	1.0269	1.005	1.0034	0.0953	0.0391	0.0203	0.0099
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta \setminus x)$)							
	1	2	-	1.0976	1.0456	1.0148	1.0084	0.0937	0.0390	0.0201	0.0099
		3	-	1.1885	1.084	1.0344	1.0183	0.1197	0.0440	0.0211	0.0101
		5	-	1.3703	1.161	1.0736	1.0381	0.2213	0.0629	0.0253	0.0113
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta \setminus x)$)							
	1	2	-	0.9612	0.9878	0.9853	0.9935	0.0857	0.0371	0.0201	0.0099
		4	-	0.8811	0.9512	0.9664	0.9837	0.0848	0.0366	0.0202	0.0099
		6	-	0.8133	0.9173	0.9481	0.9742	0.0951	0.03874	0.0211	0.0101
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta \setminus x)$)							
	1	2	0.5	0.8811	0.9512	0.9664	0.9837	0.0848	0.0366	0.0202	0.0099
		2	1	0.9227	0.9698	0.9760	0.9886	0.0767	0.0352	0.0197	0.0097
		3	0.5	0.8133	0.9173	0.9481	0.9742	0.0951	0.0387	0.0211	0.0101
		3	1	0.8517	0.9351	0.9576	0.9790	0.0822	0.0361	0.0202	0.0099
Bayes	θ	a	b	Improper distribution ($P_5(\theta \setminus x)$)							
	1	1	1	1.0067	1.0071	0.9951	0.9984	0.0842	0.0370	0.0199	0.0098
		1	2	1.0976	1.0456	1.0148	1.0084	0.0937	0.0390	0.0201	0.0099
		1	3	1.1885	1.084	1.0344	1.0183	0.1197	0.0440	0.0211	0.0101
		2	1	0.9227	0.9698	0.9760	0.9886	0.0767	0.0352	0.0197	0.0097
		2	2	1.0061	1.0068	0.9952	0.9984	0.0707	0.0343	0.0191	0.0096
		2	3	1.0894	1.0439	1.0145	1.0083	0.0787	0.0362	0.0193	0.0097
		3	1	0.8517	0.9351	0.9576	0.9790	0.0822	0.0361	0.0202	0.0099
		3	2	0.9287	0.9708	0.9764	0.9887	0.0653	0.0327	0.019	0.0096
		3	3	1.0056	1.0066	0.9953	0.9985	0.0603	0.0319	0.0184	0.0094
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta \setminus x)$)							
	1	1	-	1.0073	1.0074	0.9950	0.9984	0.1019	0.0400	0.0207	0.0100
		2	-	0.9157	0.9686	0.9755	0.9885	0.0913	0.0379	0.0205	0.0099
		3	-	0.8394	0.9327	0.9567	0.9788	0.0965	0.0388	0.0210	0.0101

Table4.7: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

Method	parameters			Estimate for θ ($\hat{\theta}$)				MWSE			
				Sample Size(n)				Sample Size(n)			
	θ	-	-	10	25	50	100	10	25	50	100
MLE	2	-	-	2.0036	2.0018	2.0041	1.9954	0.1982	0.0803	0.0390	0.0204
ME	2	-	-	2.0036	2.0018	2.0041	1.9954	0.1982	0.0803	0.0390	0.0204
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta \backslash x)$)							
	2	0.5	-	1.932	1.9724	1.9892	1.988	0.1820	0.0776	0.0383	0.0203
		1	-	1.9558	1.9822	1.9942	1.9904	0.1807	0.0773	0.0383	0.0202
		2	-	2.0034	2.0018	2.0041	1.9954	0.1797	0.0772	0.0382	0.0202
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta \backslash x)$)							
	2	2	-	2.0033	2.0018	2.004	1.9954	0.1638	0.0742	0.0375	0.0200
		3	-	2.0942	2.0402	2.0236	2.0053	0.1682	0.0750	0.0378	0.0200
		5	-	2.276	2.1171	2.0628	2.0251	0.2018	0.0811	0.0395	0.0203
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta \backslash x)$)							
	2	2	-	1.8669	1.9441	1.9746	1.9806	0.1726	0.0758	0.0378	0.0202
		4	-	1.7113	1.8721	1.9366	1.9612	0.1793	0.0770	0.0381	0.0204
		6	-	1.5797	1.8052	1.9001	1.9421	0.2056	0.0830	0.0397	0.0209
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta \backslash x)$)							
	2	2	0.5	1.7113	1.8721	1.9366	1.9612	0.1793	0.0770	0.0381	0.0204
		2	1	1.753	1.8906	1.9463	1.9661	0.1681	0.0748	0.0375	0.0202
		3	0.5	1.5797	1.8052	1.9001	1.9421	0.2056	0.0830	0.0397	0.0209
		3	1	1.6182	1.8231	1.9095	1.947	0.1901	0.0797	0.0388	0.0206
Bayes	θ	a	b	Improper distribution ($P_5(\theta \backslash x)$)							
	2	1	1	1.9124	1.9633	1.9844	1.9855	0.1676	0.0749	0.0376	0.0201
		1	2	2.0033	2.0018	2.004	1.9954	0.1638	0.0742	0.0375	0.0200
		1	3	2.0942	2.0402	2.0236	2.0053	0.1682	0.0750	0.0378	0.0200
		2	1	1.753	1.8906	1.9463	1.9661	0.1681	0.0748	0.0375	0.0202
		2	2	1.8363	1.9276	1.9655	1.9759	0.1510	0.0715	0.0367	0.0199
		2	3	1.9197	1.9647	1.9847	1.9857	0.1408	0.0695	0.0362	0.0197
		3	1	1.6182	1.8231	1.9095	1.947	0.1901	0.0797	0.0388	0.0206
		3	2	1.6951	1.8588	1.9284	1.9567	0.1637	0.0740	0.0373	0.0202
		3	3	1.772	1.8945	1.9473	1.9664	0.1432	0.0696	0.0361	0.0198
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta \backslash x)$)							
	2	1	-	2.0036	2.0018	2.0041	1.9954	0.1982	0.0803	0.0390	0.0204
		2	-	1.8215	1.9248	1.9648	1.9756	0.1797	0.0771	0.0381	0.0203
		3	-	1.6697	1.8535	1.927	1.9563	0.1922	0.0796	0.0387	0.0206

Table4.8: Shows the values for $\hat{\theta}$ under square error loss function (MWSE).

Method	parameters			Estimate for θ ($\hat{\theta}$)				MWSE			
				Sample Size(n)				Sample Size(n)			
	θ	-	-	10	25	50	100	10	25	50	100
MLE	4	-	-	4.0072	4.0037	4.0133	3.998	0.3964	0.1607	0.0825	0.0395
ME	4	-	-	4.0072	4.0037	4.0133	3.998	0.3964	0.1607	0.0825	0.0395
Bayes	θ	b_3	-	Levy distribution ($P_1(\theta \backslash x)$)							
	4	0.5	-	3.8402	3.935	3.9785	3.9806	0.3659	0.1555	0.0809	0.0392
		1	-	3.864	3.9448	3.9834	3.9831	0.3641	0.1552	0.0809	0.0392
		2	-	3.9116	3.9644	3.9933	3.9881	0.3615	0.1547	0.0808	0.0391
Bayes	θ	b	-	Gumbel type-II distribution ($P_2(\theta \backslash x)$)							
	4	2	-	3.8247	3.9266	3.9738	3.9782	0.3352	0.1499	0.0794	0.0388
		3	-	3.9156	3.9651	3.9934	3.9881	0.3293	0.1488	0.0793	0.0387
		5	-	4.0975	4.042	4.0326	4.0079	0.3299	0.1490	0.0795	0.0387
Bayes	θ	v	-	Inverse Chi-square distribution ($P_3(\theta \backslash x)$)							
	4	2	-	3.6884	3.8689	3.9444	3.9634	0.3518	0.1528	0.0800	0.0390
		4	-	3.381	3.7256	3.8685	3.9245	0.3710	0.1566	0.0806	0.0394
		6	-	3.1209	3.5925	3.7955	3.8864	0.4277	0.1696	0.0838	0.0404
Bayes	θ	α	β	Inverted Gamma distribution ($P_4(\theta \backslash x)$)							
	4	2	0.5	3.381	3.7256	3.8685	3.9245	0.3710	0.1566	0.0806	0.0394
		2	1	3.4227	3.7441	3.8781	3.9294	0.3586	0.1541	0.0799	0.0392
		3	0.5	3.1209	3.5925	3.7955	3.8864	0.4277	0.1696	0.0838	0.0404
		3	1	3.1594	3.6104	3.805	3.8913	0.4112	0.1660	0.0829	0.0402
Bayes	θ	a	b	Improper distribution ($P_5(\theta \backslash x)$)							
	4	1	1	3.7338	3.8881	3.9542	3.9683	0.3453	0.1517	0.0798	0.0390
		1	2	3.8247	3.9266	3.9738	3.9782	0.3352	0.1499	0.0794	0.0388
		1	3	3.9156	3.9651	3.9934	3.9881	0.3293	0.1488	0.0793	0.0387
		2	1	3.4227	3.7441	3.8781	3.9294	0.3586	0.1541	0.0799	0.0392
		2	2	3.506	3.7812	3.8974	3.9392	0.3362	0.1497	0.0789	0.0389
		2	3	3.5893	3.8182	3.9166	3.949	0.3174	0.1460	0.0780	0.0386
		3	1	3.1594	3.6104	3.805	3.8913	0.4112	0.1660	0.0829	0.0402
		3	2	3.2363	3.6461	3.8238	3.901	0.3803	0.1594	0.0811	0.0397
		3	3	3.3132	3.6818	3.8427	3.9107	0.3524	0.1534	0.0796	0.0392
Bayes	θ	c	-	Non-informative distribution ($P_6(\theta \backslash x)$)							
	4	1	-	4.0072	4.0037	4.0133	3.998	0.3964	0.1607	0.0825	0.0395
		2	-	3.6429	3.8497	3.9346	3.9584	0.3594	0.1542	0.0803	0.0391
		3	-	3.3393	3.7071	3.8589	3.9196	3.3393	0.1592	0.0812	0.0396

4.1 Conclusion

When we compared the estimated values for θ ($\hat{\theta}$) for the scale parameter of the Exponential distribution by using the methods in this study .We find that Mean Square Errors (MSE) and Mean weighted squared Errors (MWSE) were decreased when sample size increased in all cases. And the MSE increased in all samples sizes (n) when the true value of θ increased the same thing for MWSE. The best method is the bayes estimation according to the smallest values of MSE for all sample sizes (n) when the prior distribution is

- **Inverted Gamma distribution with ($\alpha = 3, \beta = 1$), when the true value of θ ($\theta = 0.5$) see table (4.1).**
- **Improper distribution with (a=3, b=2), when the true value of θ ($\theta = 1$) see table (4.2).**
- **Improper distribution with (a=b=3), when the true value of θ ($\theta = 2$) see table (4.3).**
- **Improper distribution with (a=b=3), when the true value of θ ($\theta = 4$) see table (4.4).**

The best method is the bayes estimation according to the smallest values of MWSE for all samples sizes (n) when the prior distribution is

- **Inverted Gamma distribution with ($\alpha = 3, \beta = 1$), & Improper distribution with (a=3, b=1), when the true value of θ ($\theta = 0.5$) see table (4.5).**
- **Improper distribution with (a=b=3), when the true value of θ ($\theta = 1$) see table (4.6).**
- **Improper distribution with (a=2, b=3), when the true value of θ ($\theta = 2$) see table (4.7).**
- **Gumbel type-II distribution with b=2 & Improper distribution with (a=1, b=3), when the true value of θ ($\theta = 4$) see table (4.8).**

4.2 Recommendations

we recommend to use the bayes estimation, to estimate scale parameter of the Exponential distribution when the prior distributions are improper distribution and Inverted Gamma distribution and Gumbel type-II distribution with the values of the parameters mentioned in our conclusions.

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Appendix-A: The posterior distribution using different Priors

1. The posterior distribution using Levy distribution as prior:

It is assumed that θ follows the Levy distribution with pdf as given below:

$$P(\theta)\,\alpha\,\sqrt{\frac{b_3}{2\pi}}\,\theta^{-\frac{3}{2}}\,\exp(-\frac{b_3}{2\theta})\quad \text{for}\quad b_3,\theta>0 \qquad \text{...(A.1)}$$

Where b_3 = hyperparameter

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ is:

$$P(\theta\backslash x)=\frac{L(\underline{x}\backslash\theta)\;P(\theta)}{\int_{\theta}L(\underline{x}\backslash\theta)\;P(\theta)d\theta} \qquad \text{...(A.2)}$$

Substituting the equation (2) and the equation (A.1) in equation (A.2), we get:

$$P_1(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta})\,[\sqrt{\frac{b_3}{2\pi}}\,\theta^{-\frac{3}{2}}\,\exp(-\frac{b_3}{2\theta})\,]}{\int_0^\infty\theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta})[\sqrt{\frac{b_3}{2\pi}}\,\theta^{-\frac{3}{2}}\,\exp(-\frac{b_3}{2\theta})\,]d\theta} \qquad \text{... (A.3)}$$

$$P_1(\theta\backslash x)=\frac{\theta^{-n-\frac{3}{2}}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))}{\int_0^\infty\theta^{-n-\frac{3}{2}}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))d\theta} \qquad \text{... (A.4)}$$

We can write $\theta^{-n+\frac{3}{2}}$ as $\theta^{-[(n+\frac{1}{2})+1]}$, and by multiplying the integral in equation (A.4) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\right)\left(\frac{\Gamma(n+\frac{1}{2})}{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}}\right)$$, where $\Gamma(.)$ is a gamma function

.Then we get,

$$P_1(\theta\backslash x)=\frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}\,\theta^{-[(n+\frac{1}{2})+1]}\,\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))}{\Gamma(n+\frac{1}{2})\;A(x;\theta)} \qquad \text{...(A.5)}$$

Where $A(x;\theta)$ equals to

$$A(x;\theta)=\int_0^\infty\frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}\,\theta^{-[(n+\frac{1}{2})+1]}\,\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2}))}{\Gamma(n+\frac{1}{2})}d\theta=1$$ .Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,...,x_n)$ is

$$P_1(\theta\backslash x)=\frac{(\sum_{i=1}^{n}x_i+\frac{b_3}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+\frac{b_3}{2})) \qquad \text{... (A.6)}$$

It means that $P_1(\theta\backslash x)\sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)}=(n+\frac{1}{2}),\beta_{(new)}=(\sum_{i=1}^{n}x_i+\frac{b_3}{2})$).

2. The posterior distribution using Gumbel type-II distribution as prior:

It is assumed that θ follows the Gumbel type-II distribution with pdf as given below: $P(\theta)\,\alpha\;a\,b\,\theta^{-(a+1)}\;\exp(-\frac{b}{\theta^a})$ for $a,b,\theta>0$ ...(A.7)

If a=1 then we get

$$P(\theta)\,\alpha\;b\,\theta^{-2}\;\exp(-\frac{b}{\theta})\quad\text{for}\quad b,\theta>0 \qquad \text{...(A.8)}$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.8) in equation (A.2), so we have

$$P_2(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta})[b\,\theta^{-2}\;\exp(-\frac{b}{\theta})]}{\int_0^\infty\theta^{-n}\exp(-\frac{\sum_{i=1}^{n}x_i}{\theta}[b\,\theta^{-2}\;\exp(-\frac{b}{\theta})]d\theta} \qquad \text{... (A.9)}$$

$$P_2(\theta\backslash x)=\frac{\theta^{-n-2}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b))}{\int_0^\infty\theta^{-n-2}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b))d\theta} \qquad \text{...(A.10)}$$

We can write θ^{-n-2} as $\theta^{-[(n+1)+1]}$, and by multiplying the integral in equation (A.10) by the quantity which equals to

$$\left(\frac{(\sum_{i=1}^{n}x_i+b)^{(n+1)}}{\Gamma(n+1)}\right)\left(\frac{\Gamma(n+1)}{(\sum_{i=1}^{n}x_i+b)^{(n+1)}}\right)$$, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_2(\theta\backslash x)=\frac{(\sum_{i=1}^{n}x_i+b)^{(n+1)}\;\theta^{-[(n+1)+1]}\;\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b))}{\Gamma(n+1)\;\;B(x;\theta)} \qquad \text{... (A.11)}$$

Where $B(x;\theta)$ equals to

$B(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^{(n+1)}\ \theta^{-[(n+1)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+1)}d\theta=1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,...,x_n)$ is

$$P_2(\theta\backslash x)=\frac{(\sum_{i=1}^{n} x_i+b)^{(n+1)}\ \theta^{-[(n+1)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+1)} \quad ...(A.12)$$

It means that $P_2(\theta\backslash x)\sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)}=(n+1),\beta_{(new)}=(\sum_{i=1}^{n} x_i+b)$).

3. The posterior distribution using Inverse Chi-square distribution as prior:

It is assumed that θ follows the Inverse Chi-square distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{1}{2^{\frac{v}{2}}}\theta^{-\frac{v}{2}-1}\exp(-\frac{1}{2\theta}) \quad \text{for } v,\theta>0 \quad ...(A.13)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.13) in equation (A.2), so we have

$$P_3(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\frac{1}{2^{\frac{v}{2}}}\theta^{-\frac{v}{2}-1}\exp(-\frac{1}{2\theta})]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\frac{1}{2^{\frac{v}{2}}}\theta^{-\frac{v}{2}-1}\exp(-\frac{1}{2\theta})]d\theta} \quad ...(A.14)$$

$$P_3(\theta\backslash x)=\frac{\theta^{-[(n+\frac{v}{2})+1]}\ e^{-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2})}}{\int_0^\infty \theta^{-[(n+\frac{v}{2})+1]}\ e^{-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2})}d\theta} \quad ...(A.15)$$

By multiplying the integral in equation (A.15) by the quantity which equals to

$(\frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})})(\frac{\Gamma(n+\frac{v}{2})}{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}})$, where $\Gamma(.)$ is a gamma function. Then we get,

$$P_3(\theta\backslash x)=\frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}\ \theta^{-[(n-\frac{v}{2})+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))}{\Gamma(n+\frac{v}{2})\ \ C(x;\theta)} \quad ...(A.16)$$

Where $C(x;\theta)$ equals to

$C(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}\ \theta^{-[(n+\frac{v}{2})+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))}{\Gamma(n+\frac{v}{2})}d\theta=1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,...,x_n)$ is

$$P_3(\theta\backslash x)=\frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\ \theta^{-[(n+\frac{v}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2})) \quad ...(A.17)$$

It means that $P_3(\theta\backslash x)$)~Inverted Gamma distribution with new parameters ($\alpha_{(new)}=(n+\frac{v}{2}),\beta_{(new)}=(\sum_{i=1}^{n} x_i+\frac{1}{2})$).

4. The posterior distribution using Inverted Gamma distribution as prior:

It is assumed that θ follows the Inverted Gamma distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{\beta^{\alpha}}{\Gamma\alpha}\theta^{-(\alpha+1)}\exp(-\frac{\beta}{\theta}) \quad \text{for } \alpha,\beta,\theta>0 \quad ...(A.18)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,...,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.18) in equation (A.2), so we have

$$P_4(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\frac{\beta^{\alpha}}{\Gamma\alpha}\theta^{-(\alpha+1)}\exp(-\frac{\beta}{\theta})]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^{n} x_i}{\theta})[\frac{\beta^{\alpha}}{\Gamma\alpha}\theta^{-(\alpha+1)}\exp(-\frac{\beta}{\theta})]d\theta} \quad ...(A.19)$$

$$P_4(\theta\backslash x)=\frac{\theta^{-[(n+\alpha)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\int_0^\infty \theta^{-[(n+\alpha)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))d\theta} \quad ...(A.20)$$

By multiplying the integral in equation (A.20) by the quantity which equals to

$(\frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)}}{\Gamma(n+\alpha)})(\frac{\Gamma(n+\alpha)}{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)}})$, where $\Gamma(.)$ is a gamma function. Then we get,

$$P_4(\theta\backslash x)=\frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)}\ \theta^{-[(n+\alpha)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)\ \ D(x;\theta)} \quad ...(A.21)$$

Where $D(x;\theta)$ equals to

$$D(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^n x_i+\beta)^{(n+\alpha)}\ \theta^{-[(n+\alpha)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\beta))}{\Gamma(n+\alpha)}d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,\ldots,x_n)$ is

$$P_4(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i+\beta)^{(n+\alpha)}\ \theta^{-[(n+\alpha)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+\beta))}{\Gamma(n+\alpha)} \quad \ldots(A.22)$$

It means that $P_4(\theta\backslash x)\sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)}=(n+\alpha),\ \beta_{(new)}=(\sum_{i=1}^n x_i+\beta)$).

5. The posterior distribution using improper distribution as prior:

It is assumed that θ follows the improper distribution with pdf as given below:

$$P(\theta)\ \alpha\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta}) \quad \text{for } b,\ \theta>0 \ \text{ and } -\infty<a<\infty \quad \ldots(A.23)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,\ldots,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.22) in equation (A.2), so we have

$$P_5(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta})\ ]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\ \theta^{-(a+1)}\ \exp(-\frac{b}{\theta})\ ]d\theta} \quad \ldots(A.24)$$

$$P_5(\theta\backslash x)=\frac{\theta^{-[(n+a)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+b))}{\int_0^\infty \theta^{-[(n+a)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+b))d\theta} \quad \ldots(A.25)$$

By multiplying the integral in equation (A.25) by the quantity which equals to

$$(\frac{(\sum_{i=1}^n x_i+b)^{(n+a)}}{\Gamma(n+a)})\ (\frac{\Gamma(n+a)}{(\sum_{i=1}^n x_i+b)^{(n+a)}})$$, where $\Gamma(.)$ is a gamma function. Then we get,

$$P_5(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i+b)^{(n+a)}\ \theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+b))}{\Gamma(n+a)\quad E(x;\theta)} \quad \ldots(A.26)$$

Where $E(x;\theta)$ equals to

$$E(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^n x_i+b)^{(n+a)}\ \theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+b))}{\Gamma(n+a)}d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,\ldots,x_n)$ is

$$P_5(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i+b)^{(n+a)}\ \theta^{-[(n+a)+1]}\ \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i+b))}{\Gamma(n+a)} \quad \ldots(A.27)$$

It means that $P_5(\theta\backslash x)\sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)}=(n+a),\ \beta_{(new)}=(\sum_{i=1}^n x_i+b)$).

6. The posterior distribution using Non-informative distribution as prior:

It is assumed that θ follows the non-informative distribution with pdf as given below:

$$P(\theta)\ \alpha\ \frac{1}{\theta^c} \quad \text{for } \theta,c>0 \quad \ldots(A.28)$$

Then the posterior distribution of given the data $\underline{x}=(x_1,x_2,\ldots,x_n)$ according to the equation (A.2), we get it by substituting the equation (2) and the equation (A.28) in equation (A.2), so we have

$$P_6(\theta\backslash x)=\frac{\theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\ \theta^{-c}\ ]}{\int_0^\infty \theta^{-n}\exp(-\frac{\sum_{i=1}^n x_i}{\theta})[\ \theta^{-c}\ ]d\theta} \quad \ldots(A.29)$$

$$P_6(\theta\backslash x)=\frac{\theta^{-(n+c)}\exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\int_0^\infty \theta^{-(n+c)}\exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)d\theta} \quad \ldots(A.30)$$

We can write $\theta^{-(n+c)}$ as $\theta^{-[(n+c-1)+1]}$, and by multiplying the integral in equation (A.30), by the quantity which equals to

$$(\frac{(\sum_{i=1}^n x_i)^{(n+c-1)}}{\Gamma(n+c-1)})\ (\frac{\Gamma(n+c-1)}{(\sum_{i=1}^n x_i)^{(n+c-1)}})$$, where $\Gamma(.)$ is a gamma function. Then we get

$$P_6(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\ \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\Gamma(n+c-1)\quad F(x;\theta)} \quad \ldots(A.31)$$

Where $F(x;\theta)$ equals to

$$F(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^n x_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\ \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\Gamma(n+c-1)}d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{x}=(x_1,x_2,\ldots,x_n)$ is

$$P_6(\theta\backslash x)=\frac{(\sum_{i=1}^n x_i)^{(n+c-1)}\ \theta^{-[(n+c-1)+1]}\ \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\Gamma(n+c-1)} \quad \ldots(A.32)$$

It means that $P_6(\theta\backslash x)\sim$ Inverted Gamma distribution with new parameters ($\alpha_{(new)}=(n+c-1),\ \beta_{(new)}=(\sum_{i=1}^n x_i)$).

Appendix-B: The following is the derivation of these estimators under the squared error loss function.

1. **The squared error loss function**

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_1(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2$, the risk function is:

$$R(\hat{\theta}-\theta)=E[L_1(\hat{\theta},\theta)] \quad ...(B.1)$$

$$R(\hat{\theta}-\theta)=\int_\theta L_1(\hat{\theta},\theta)\,P(\theta\backslash x)\,d\theta$$

$$R(\hat{\theta}-\theta)=\int_\theta(\hat{\theta}-\theta)^2\,P(\theta\backslash x)\,d\theta \Rightarrow R(\hat{\theta}-\theta)=\int_\theta(\hat{\theta}^2-2\hat{\theta}\theta+\theta^2)\,P(\theta\backslash x)\,d\theta$$

$$R(\hat{\theta}-\theta)=\hat{\theta}^2\int P(\theta\backslash x)\,d\theta-2\hat{\theta}\int\theta P(\theta\backslash x)\,d\theta+\int\theta^2 P(\theta\backslash x)\,d\theta \Rightarrow$$

$$R(\hat{\theta}-\theta)=\hat{\theta}^2-2\hat{\theta}E(\theta\backslash x)+E(\theta^2\backslash x) \quad ...(B.2)$$

Let $\frac{\partial}{\partial\hat{\theta}}R(\hat{\theta}-\theta)=0$, we get Bayes estimator of θ denoted by $\hat{\theta}_{Bayes}$ for the above prior as follows

$$\hat{\theta}_{SE}=E(\theta\backslash x)=\int_0^\infty\theta P(\theta\backslash x)\,d\theta \quad ...(B.3)$$

1.1 **Bayes estimation using Levy distribution as prior**

To obtain the Bayes' estimator under Levy distribution as prior. Substituting the equation (A.6) in equation (B.3), we get:

$$\hat{\theta}_{SE1}=\int_0^\infty\theta P_1(\theta\backslash x)\,d\theta$$

$$\hat{\theta}_{SE1}=\int_0^\infty\theta\,\frac{(\Sigma_{i=1}^n x_i+\frac{b_2}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n x_i+\frac{b_2}{2}))d\theta \quad ...(B.4)$$

$$\hat{\theta}_{SE1}=\int_0^\infty\frac{(\Sigma_{i=1}^n x_i+\frac{b_2}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{1}{2})+1]+1}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n x_i+\frac{b_2}{2}))d\theta \quad ...(B.5)$$

For the equation (45), we can write $-[(n+\frac{1}{2})+1]+1=-[n+\frac{1}{2}+1-1]=-[(n-\frac{1}{2})+1]$

That is $(n+\frac{1}{2})=n+\frac{1}{2}+\frac{1}{2}-\frac{1}{2}=-[(n-\frac{1}{2})+1]$.

By multiplying the integral in equation (B.5) by the quantity which equals to $A1=(\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-\frac{1}{2})})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$\hat{\theta}_{SE1}=A1\int_0^\infty\frac{(\Sigma_{i=1}^n x_i+\frac{b_2}{2})^{(n-\frac{1}{2})+1}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n-\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n x_i+\frac{b_2}{2}))\,d\theta \quad ...(B.6)$$

Then, we have

$$\hat{\theta}_{SE1}=\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+\frac{1}{2})}(\Sigma_{i=1}^n x_i+\frac{b_2}{2})\,(A2(x;\theta)) \quad ...(B.7)$$

Where $A2(x;\theta)$ equals to

$$A2(x;\theta)=\int_0^\infty\frac{(\Sigma_{i=1}^n x_i+\frac{b_2}{2})^{(n-\frac{1}{2})}}{\Gamma(n-\frac{1}{2})}\theta^{-[(n-\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n x_i+\frac{b_2}{2}))\,d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE1}=\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+\frac{1}{2})}(\Sigma_{i=1}^n x_i+\frac{b_2}{2}) \quad ,n\,\&\,b_2>0 \quad ...(B.8)$$

1.2 **Bayes estimation using Gumbel type-II distribution as prior:**

To obtain the Bayes' estimator under the Gumbel type-II distribution as prior distribution. Substituting the equation (A.12) in equation (B.3), we get:

$$\hat{\theta}_{SE2}=\int_0^\infty\theta P_2(\theta\backslash x)\,d\theta$$

$$\hat{\theta}_{SE2}=\int_0^\infty\theta\,\frac{(\Sigma_{i=1}^n x_i+b)^{(n+1)}\;\theta^{-[(n+1)+1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n x_i+b))}{\Gamma(n+1)}d\theta \quad ...(B.9)$$

$$\hat{\theta}_{SE2}=\int_0^\infty\frac{(\Sigma_{i=1}^n x_i+b)^{(n+1)}\;\theta^{-[(n+1)+1-1]}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n x_i+b))}{\Gamma(n+1)}d\theta \quad ...(B.10)$$

For the equation (B.10), we can write $-[(n+1)+1-1]=-[n+1]$ And by multiplying the integral in equation (B.10) by the quantity which equals to $B_1=(\frac{\Gamma(n)}{\Gamma(n)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$\hat{\theta}_{SE2}=B_1\int_0^\infty\frac{(\Sigma_{i=1}^n x_i+b)^{(n+1)}\;\theta^{-(n+1)}\exp(-\frac{1}{\theta}(\Sigma_{i=1}^n x_i+b))}{\Gamma(n+1)}d\theta \quad ...(B.11)$$

Then we have

$$\hat{\theta}_{SE2}=\frac{\Gamma(n)}{\Gamma(n+1)}(\Sigma_{i=1}^n x_i+b)\,(B2(x;\theta)) \quad ...(B.12)$$

Where $B2(x;\theta)$ equals to

$$B2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^n \ \theta^{-(n+1)} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n)} d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE2}=\frac{\Gamma(n)}{\Gamma(n+1)}(\sum_{i=1}^{n} x_i+b) \quad , n \,\&\, b>0 \qquad \text{... (B.13)}$$

1.3 Bayes estimation using Inverse chi-squared distribution as prior:

To obtain the Bayes' estimator under inverse chi-squared distribution as prior. Substituting the equation (A.16) in equation (B.3), we get:

$$\hat{\theta}_{SE3}=\int_0^\infty \theta P_3(\theta\setminus x)\,d\theta$$

$$\hat{\theta}_{SE3}=\int_0^\infty \theta \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta \qquad \text{... (B.14)}$$

$$\hat{\theta}_{SE3}=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2})+1]+1} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta \qquad \text{... (B.15)}$$

For the equation (B.15), we can write $-[((n+\frac{v}{2})+1)-1]=-[(n+\frac{v}{2}-1)+1]$ And by multiplying the integral in equation (B.15) by the quantity which equals to $C1=(\frac{\Gamma(n+\frac{v}{2}-1)}{\Gamma(n+\frac{v}{2}-1)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$\hat{\theta}_{SE3}=C1\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})+1-1}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2}-1])+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta \qquad \text{... (B.16)}$$

Then we have

$$\hat{\theta}_{SE3}=\frac{\Gamma(n+\frac{v}{2}-1)}{\Gamma(n+\frac{v}{2})}(\sum_{i=1}^{n} x_i+\frac{1}{2})\ (C2(x;\theta)) \qquad \text{... (B.17)}$$

Where $C2(x;\theta)$ equals to

$$C2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2}-1)}}{\Gamma(n+\frac{v}{2})} \theta^{-[(n+\frac{v}{2}-1)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE3}=\frac{\Gamma(n+\frac{v}{2}-1)}{\Gamma(n+\frac{v}{2})}(\sum_{i=1}^{n} x_i+\frac{1}{2}) \quad , n \,\&\, v>0 \qquad \text{... (B.18)}$$

1.4 Bays estimation using Inverted gamma distribution as prior:

To obtain the Bayes' estimator under the inverted gamma distribution as prior. Substituting the equation (A.21) in equation (B.3), we get:

$$\hat{\theta}_{SE4}=\int_0^\infty \theta P_4(\theta\setminus x)\,d\theta$$

$$\hat{\theta}_{SE4}=\int_0^\infty \theta \frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)} \ \theta^{-[(n+\alpha)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)} d\theta \qquad \text{... (B.19)}$$

$$\hat{\theta}_{SE4}=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)} \ \theta^{-[(n+\alpha)+1]+1} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)} \qquad \text{... (B.20)}$$

For the equation (B.20), we can write $-[(n+\alpha)+1]+1=-[(n+\alpha-1)+1]$ And by multiplying the integral in equation (B.20) by the quantity which equals to $D1=(\frac{\Gamma(n+\alpha-1)}{\Gamma(n+\alpha-1)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$\hat{\theta}_{SE4}=D1\int_0^\infty \frac{\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)-1+1} \theta^{-[(n+\alpha-1)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)} d\theta \qquad \text{... (B.21)}$$

Then we have

$$\hat{\theta}_{SE4}=\frac{\Gamma(n+\alpha-1)}{\Gamma(n+\alpha)}(\sum_{i=1}^{n} x_i+\beta)\,(D2(x;\theta)) \qquad \text{... (B.22)}$$

Where $D2(x,\theta)$ equals to

$$D2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha-1)} \theta^{-[(n+\alpha-1)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha-1)} d\theta=1.$$ Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SE4}=\frac{\Gamma(n+\alpha-1)}{\Gamma(n+\alpha)}(\sum_{i=1}^{n} x_i+\beta) \qquad n,\beta,\alpha>0 \qquad \text{... (B.23)}$$

1.5 Bayes estimation using improper distribution as prior:

To obtain the Bayes' estimator under improper distribution as prior. Substituting the equation (A.26) in equation (B.3), we get:

$$\hat{\theta}_{SE5}=\int_0^\infty \theta P_5(\theta\setminus x)\,d\theta$$

$$\hat{\theta}_{SE5}=\int_0^\infty \theta \frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)} \ \theta^{-[(n+a)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+a)} d\theta \qquad \text{... (B.24)}$$

$$\hat{\theta}_{SS5} = \int_0^\infty \frac{(\sum_{i=1}^n x_i + b)^{(n+a)}\, \theta^{-[(n+a)+1]+1} \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i + b))}{\Gamma(n+a)}\, d\theta \quad \text{... (B.25)}$$

For the equation (B.25), we can write $-[(n+a)+1]+1 = -[(n+a-1)+1]$ And by multiplying the integral in equation (B.25) by the quantity which equals to $E1 = (\frac{\Gamma(n+a-1)}{\Gamma(n+a-1)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$\hat{\theta}_{SS5} = E1 \int_0^\infty \frac{(\sum_{i=1}^n x_i + b)^{(n+a)}\, \theta^{-[(n+a)+1]+1} \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i + b))}{\Gamma(n+a)}\, d\theta \quad \text{... (B.26)}$$

Then we have

$$\hat{\theta}_{SS5} = \frac{\Gamma(n+a-1)}{\Gamma(n+a)} (\sum_{i=1}^n x_i + b)\,(E2(x;\theta)) \quad \text{... (B.27)}$$

Where $E2(x;\theta)$ equals to

$E2(x;\theta) = \int_0^\infty \frac{(\sum_{i=1}^n x_i + b)^{(n+a-1)}\, \theta^{-[(n+a-1)+1]} \exp(-\frac{1}{\theta}(\sum_{i=1}^n x_i + b))}{\Gamma(n+a-1)}\, d\theta = 1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SS5} = \frac{\Gamma(n+a-1)}{\Gamma(n+a)} (\sum_{i=1}^n x_i + b) \qquad n, b, a > 0 \quad \text{... (B.28)}$$

1.6 <u>Bayes estimation using non-informative distribution as prior:</u>

To obtain the Bayes' estimator under non informative distribution as prior. Substituting the equation (A.32) in equation (B.3), we get:

$$\hat{\theta}_{SS6} = \int_0^\infty \theta P_6(\theta \backslash x)\, d\theta$$

$$\hat{\theta}_{SS6} = \int_0^\infty \theta \frac{(\sum_{i=1}^n x_i)^{(n+c-1)}\, \theta^{-[(n+c-1)+1]} \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\Gamma(n+c-1)}\, d\theta \quad \text{... (B.29)}$$

$$\hat{\theta}_{SS6} = \int_0^\infty \frac{(\sum_{i=1}^n x_i)^{(n+c-1)}\, \theta^{-[(n+c-1)+1]+1} \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\Gamma(n+c-1)}\, d\theta \quad \text{... (B.30)}$$

For the equation (B.30), we can write $-[(n+c-1)+1]+1 = -[(n+c-2)+1]$ And by multiplying the integral in equation (B.30) by the quantity which equals to $F1 = (\frac{\Gamma(n+c-2)}{\Gamma(n+c-2)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$\hat{\theta}_{SS6} = F1 \int_0^\infty \frac{(\sum_{i=1}^n x_i)^{(n+c-1)+1-1}\, \theta^{-[(n+c-2)+1]} \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\Gamma(n+c-1)}\, d\theta \quad \text{... (B.31)}$$

Then we have

$$\hat{\theta}_{SS6} = \frac{\Gamma(n+c-2)}{\Gamma(n+c-1)} (\sum_{i=1}^n x_i)\,(F2(x;\theta)) \quad \text{... (B.32)}$$

Where $F2(x;\theta)$ equals to

$F2(x;\theta) = \int_0^\infty \frac{(\sum_{i=1}^n x_i)^{(n+c-2)}\, \theta^{-[(n+c-2)+1]} \exp(-\frac{1}{\theta}\sum_{i=1}^n x_i)}{\Gamma(n+c-2)}\, d\theta = 1$. Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$\hat{\theta}_{SS6} = \frac{\Gamma(n+c-2)}{\Gamma(n+c-1)} (\sum_{i=1}^n x_i) \qquad n, c > 0 \quad \text{... (B.33)}$$

<u>Appendix-C</u>: The following is the derivation of these estimators under the weighted squared error loss function.

1. <u>The weighted squared error loss function</u>

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta}$, the risk function is:

$$R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = E[L_2(\hat{\theta}, \theta)] \quad \text{...(C.1)}$$

$$R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = \int_\theta L_2(\hat{\theta}, \theta)\, P(\theta \backslash x)\, d\theta \;\Rightarrow\; R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = \int_\theta \frac{(\hat{\theta} - \theta)^2}{\theta} P(\theta \backslash x)\, d\theta$$

$$R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = \int_\theta \frac{(\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2)}{\theta} P(\theta \backslash x)\, d\theta$$

$$R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = \int_\theta (\frac{\hat{\theta}^2}{\theta} - 2\hat{\theta} + \theta) P(\theta \backslash x)\, d\theta$$

$$R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = \hat{\theta}^2 \int_\theta \frac{1}{\theta} P(\theta \backslash x)\, d\theta - 2\hat{\theta} \int_\theta P(\theta \backslash x)\, d\theta + \int_\theta \theta P(\theta \backslash x)\, d\theta$$

$$R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = \hat{\theta}^2 E(\frac{1}{\theta} \backslash x) - 2\hat{\theta} + E(\theta \backslash x) \quad \text{... (C.2)}$$

Let $\frac{\partial}{\partial \hat{\theta}} R_2(\frac{(\hat{\theta} - \theta)^2}{\theta}) = 0$, we get Bayes estimator of θ denoted by $\hat{\theta}_{wss}$ for the above prior as follows

$$\hat{\theta}_{wss} = \frac{1}{E(\frac{1}{\theta} \backslash x)} = \frac{1}{\int_0^\infty \frac{1}{\theta} P(\theta \backslash x)\, d\theta} \quad \text{... (C.3)}$$

1.1 <u>Bayes estimation using Levy distribution as prior</u>

To obtain the Bayes' estimator under Levy distribution as prior. Substituting the equation (A.6) in the integral in equation (C.3), we get:

$$E(\frac{1}{\theta} \backslash x) = \int_0^\infty \frac{1}{\theta} P_1(\theta \backslash x)\, d\theta$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}\frac{(\sum_{i=1}^{n} x_i+\frac{b_2}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{1}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{b_2}{2}))d\theta \quad \ldots (C.4)$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{b_2}{2})^{(n+\frac{1}{2})}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{1}{2})+1]-1}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{b_2}{2}))d\theta \quad \ldots (C.5)$$

For the equation (C.5), we can write $-[(n+\frac{1}{2})+1]-1=-[n+\frac{1}{2}+1+1]=-[(n+\frac{3}{2})+1]$. By multiplying the integral in equation (C.5) by the quantity which equals to $A1=(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+\frac{3}{2})})$, where $\Gamma(.)$ is a gamma function .

Then, we have

$$E(\frac{1}{\theta}\backslash x)=A1\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{b_2}{2})^{(n+\frac{1}{2})+1-1}}{\Gamma(n+\frac{1}{2})}\theta^{-[(n+\frac{3}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{b_2}{2}))d\theta \quad \ldots (C.6)$$

then, we have

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+\frac{1}{2})}(\sum_{i=1}^{n} x_i+\frac{b_2}{2})^{}\ (A2(x;\theta)) \quad \ldots (C.7)$$

Where $A2(x;\theta)$ equals to

$$A2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{b_2}{2})^{(n+\frac{3}{2})}}{\Gamma(n+\frac{3}{2})}\theta^{-[(n+\frac{3}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{b_2}{2}))d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+\frac{1}{2})(\sum_{i=1}^{n} x_i+\frac{b_3}{2})} \quad ,n\ \&\ b_2>0 \quad \ldots (C.8)$$

Substituting the equation (C.8) in equation (C.3), we get:

$$\hat{\theta}_{WSE1}=\frac{\Gamma(n+\frac{1}{2})(\sum_{i=1}^{n} x_i+\frac{b_2}{2})}{\Gamma(n+\frac{3}{2})} \quad ,n\ \&\ b_2>0 \quad \ldots (C.9)$$

1.2 <u>Bayes estimation using Gumbel type-II distribution as prior:</u>

To obtain the Bayes' estimator under the Gumbel type-II distribution as prior. Substituting the equation (A.12) in the integral in equation (C.3), we get:

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}P_2(\theta\backslash x)\,d\theta$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}\frac{(\sum_{i=1}^{n} x_i+b)^{(n+1)}\ \theta^{-[(n+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+1)}d\theta \quad \ldots (C.10)$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^{(n+1)+1-1}\ \theta^{-[(n+1+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+1)}d\theta \quad \ldots (C.11)$$

For the equation (C.11), we can write $-[(n+1+1)+1]=-[(n+2)+1]$. And by multiplying the integral in equation (C.11) by the quantity which equals to $B_1=(\frac{\Gamma(n+2)}{\Gamma(n+2)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$E(\frac{1}{\theta}\backslash x)=B_1\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^{(n+2)-1}\ \theta^{-[(n+2)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+1)}d\theta \quad \ldots (C.12)$$

Then we have

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+2)}{\Gamma(n+1)(\sum_{i=1}^{n} x_i+b)}\ (B2(x;\theta)) \quad \ldots (C.13)$$

Where $B2(x;\theta)$ equals to

$$B2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^{(n+2)}\ \theta^{-[(n+2)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+2)}d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+2)}{\Gamma(n+1)(\sum_{i=1}^{n} x_i+b)} \quad ,n\ \&\ b>0 \quad \ldots (C.14)$$

Substituting the equation (C.14) in equation (C.3), we get:

$$\hat{\theta}_{WSE2}=\frac{\Gamma(n+1)(\sum_{i=1}^{n} x_i+b)}{\Gamma(n+2)} \quad ,n\ \&\ b>0 \quad \ldots (C.15)$$

1.3 <u>Bayes estimation using Inverse chi-squared distribution as prior:</u>

To obtain the Bayes ' estimator under inverse chi-squared distribution as prior. Substituting the equation (A.16) in the integral in equation (C.3), we get:

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}P_3(\theta\backslash x)\,d\theta$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}\frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\theta^{-[(n+\frac{v}{2})+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta \quad \ldots (C.16)$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})}}{\Gamma(n+\frac{v}{2})}\theta^{-[(n+\frac{v}{2})+1]-1}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta \quad \ldots (C.17)$$

For the equation (C.17), we can write $-[((n+\frac{v}{2})+1)-1=-[(n+\frac{v}{2}+1)+1]$ And by multiplying the integral in equation (C.17) by the quantity which equals to $C1=(\frac{\Gamma(n+\frac{v}{2}+1)}{\Gamma(n+\frac{v}{2}+1)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$E(\frac{1}{\theta}\backslash x)=C1\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2})+1-1}}{\Gamma(n+\frac{v}{2})}\theta^{-[(n+\frac{v}{2}+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta \quad \dots (C.18)$$

Then we have

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+\frac{v}{2}+1)}{\Gamma(n+\frac{v}{2})(\sum_{i=1}^{n} x_i+\frac{1}{2})}(C2(x;\theta)) \quad \dots (C.19)$$

Where $C2(x;\theta)$ equals to

$$C2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\frac{1}{2})^{(n+\frac{v}{2}+1)}}{\Gamma(n+\frac{v}{2}+1)}\theta^{-[(n+\frac{v}{2}+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\frac{1}{2}))\,d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+\frac{v}{2}+1)}{\Gamma(n+\frac{v}{2})(\sum_{i=1}^{n} x_i+\frac{1}{2})} \qquad n, v>0 \quad \dots (C.20)$$

Substituting the equation (C.20) in equation (C.3), we get:

$$\hat{\theta}_{wsz} = \frac{\Gamma(n+\frac{v}{2})(\sum_{i=1}^{n} x_i+\frac{1}{2})}{\Gamma(n+\frac{v}{2}+1)} \qquad n, v>0 \quad \dots (C.21)$$

1.4 Bayes estimation using Inverted gamma distribution as prior:

To obtain the Bayes' estimator under the inverted gamma distribution as prior. Substituting the equation (A.21) in the integral in equation (C.3), we get:

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}P_4(\theta\backslash x)\,d\theta$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}\frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)}\theta^{-[(n+\alpha)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)}d\theta \quad \dots (C.22)$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)}\theta^{-[(n+\alpha)+1]-1}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)} \quad \dots (C.23)$$

For the equation (C.23), we can write $-[(n+\alpha)+1]-1=-[(n+\alpha+1)+1]$ And by multiplying the integral in equation (C.23) by the quantity which equals $D1=(\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+1)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$E(\frac{1}{\theta}\backslash x)=D1\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha)-1+1}\theta^{-[(n+\alpha+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha)}d\theta \quad \dots (C.24)$$

Then we have

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha)(\sum_{i=1}^{n} x_i+\beta)}(D2(x;\theta)) \quad \dots (C.25)$$

Where $D2(x;\theta)$ equals to

$$D2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+\beta)^{(n+\alpha+1)}\theta^{-[(n+\alpha+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+\beta))}{\Gamma(n+\alpha+1)}d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha)(\sum_{i=1}^{n} x_i+\beta)} \qquad n,\beta,\alpha>0 \quad \dots (C.26)$$

Substituting the equation (C.26) in equation (C.3), we get:

$$\hat{\theta}_{wsz4} = \frac{\Gamma(n+\alpha)(\sum_{i=1}^{n} x_i+\beta)}{\Gamma(n+\alpha+1)} \qquad n,\beta,\alpha>0 \quad \dots (C.27)$$

1.5 Bayes estimation using improper distribution as prior:

To obtain the Bayes' estimator under improper distribution as prior . Substituting the equation (A.26) in the integral in equation (C.3), we get:

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}P_5(\theta\backslash x)\,d\theta$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}\frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)}\theta^{-[(n+a)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+a)}d\theta \quad \dots (C.28)$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)}\theta^{-[(n+a)+1]-1}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+a)}d\theta \quad \dots (C.29)$$

For the equation (C.29), we can write $-[(n+a)+1]-1=-[(n+a+1)+1]$ And by multiplying the integral in equation (C.29) by the quantity which equals to $E1=(\frac{\Gamma(n+a+1)}{\Gamma(n+a+1)})$, where $\Gamma(.)$ is a gamma function.

Then, we have

$$E(\frac{1}{\theta}\backslash x)=E1\int_0^\infty \frac{(\sum_{i=1}^{n} x_i+b)^{(n+a)+1-1}\theta^{-[(n+a+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n} x_i+b))}{\Gamma(n+a)}d\theta \quad \dots (C.30)$$

Then we have

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+a+1)}{\Gamma(n+a)(\sum_{i=1}^{n}x_i+b)}(E_2(x;\theta)) \quad \ldots (C.31)$$

Where $E_2(x;\theta)$ equals to

$$E_2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n}x_i+b)^{(n+a+1)}\theta^{-[(n+a+1)+1]}\exp(-\frac{1}{\theta}(\sum_{i=1}^{n}x_i+b))}{\Gamma(n+a+1)}d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+a+1)}{\Gamma(n+a)(\sum_{i=1}^{n}x_i+b)} \qquad n,b,a>0 \quad \ldots (C.32)$$

Substituting the equation (C.32) in equation (C.3), we get:

$$\hat{\theta}_{WSE5}=\frac{\Gamma(n+a)(\sum_{i=1}^{n}x_i+b)}{\Gamma(n+a+1)} \qquad n,b,a>0 \quad \ldots (C.33)$$

1.6 <u>Bayes estimation using non-informative distribution as prior:</u>

To obtain the Bayes' estimator under non infortmative distribution as prior . Substituting the equation (A.32) in the integral in equation (C.3), we get:

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}P_6(\theta\backslash x)\,d\theta$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{1}{\theta}\frac{(\sum_{i=1}^{n}x_i)^{(n+c-1)}\theta^{-[(n+c-1)+1]}\exp(-\frac{1}{\theta}\sum_{i=1}^{n}x_i)}{\Gamma(n+c-1)}d\theta \quad \ldots (C.34)$$

$$E(\frac{1}{\theta}\backslash x)=\int_0^\infty \frac{(\sum_{i=1}^{n}x_i)^{(n+c-1)}\theta^{-[(n+c-1)+1]-1}\exp(-\frac{1}{\theta}\sum_{i=1}^{n}x_i)}{\Gamma(n+c-1)}d\theta \quad \ldots (C.35)$$

For the equation (C.35), we can write $-[(n+c-1)+1]-1=-[(n+c)+1]$ And by multiplying the integral in equation (C.35) by the quantity which equals to $F_1=(\frac{\Gamma(n+c)}{\Gamma(n+c)})$, where $\Gamma(.)$is a gamma function.

Then, we have

$$E(\frac{1}{\theta}\backslash x)=F_1\int_0^\infty \frac{(\sum_{i=1}^{n}x_i)^{(n+c-1)+1-1}\theta^{-[(n+c-2)+1]}\exp(-\frac{1}{\theta}\sum_{i=1}^{n}x_i)}{\Gamma(n+c-1)}d\theta \quad \ldots (C.36)$$

Then we have

$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+c)}{\Gamma(n+c-1)(\sum_{i=1}^{n}x_i)}(F_2(x;\theta)) \quad \ldots (C.37)$$

Where $F_2(x;\theta)$ equals to

$$F_2(x;\theta)=\int_0^\infty \frac{(\sum_{i=1}^{n}x_i)^{(n+c)}\theta^{-[(n+c)+1]}\exp(-\frac{1}{\theta}\sum_{i=1}^{n}x_i)}{\Gamma(n+c)}d\theta=1.$$

Be the integral of the pdf of the Inverted Gamma distribution. Then we get the Bayes estimator of θ as the following formula:

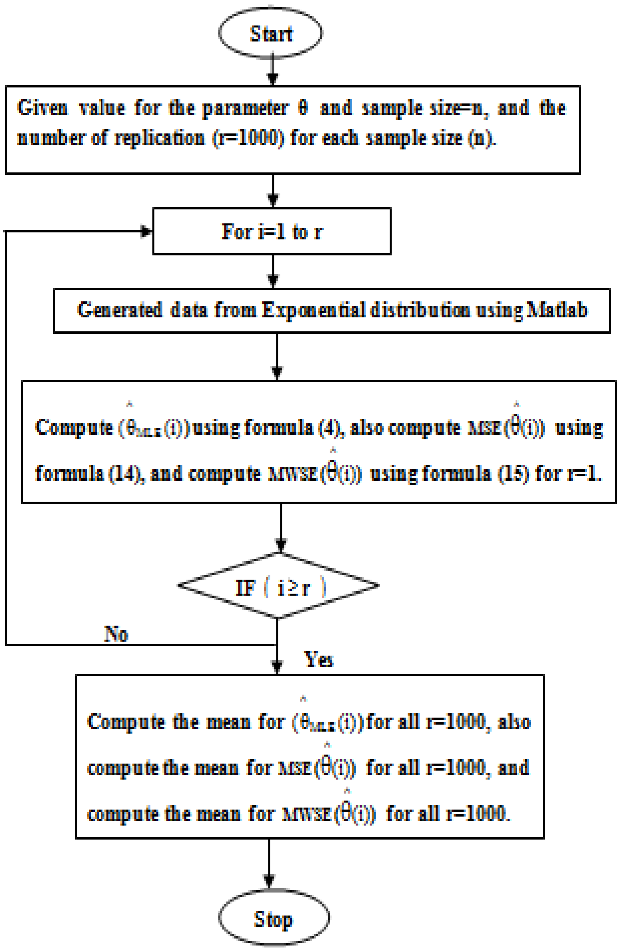
$$E(\frac{1}{\theta}\backslash x)=\frac{\Gamma(n+c)}{\Gamma(n+c-1)(\sum_{i=1}^{n}x_i)} \qquad n,c>0 \quad \ldots (C.38)$$

Substituting the equation (C.38) in equation (C.3), we get:

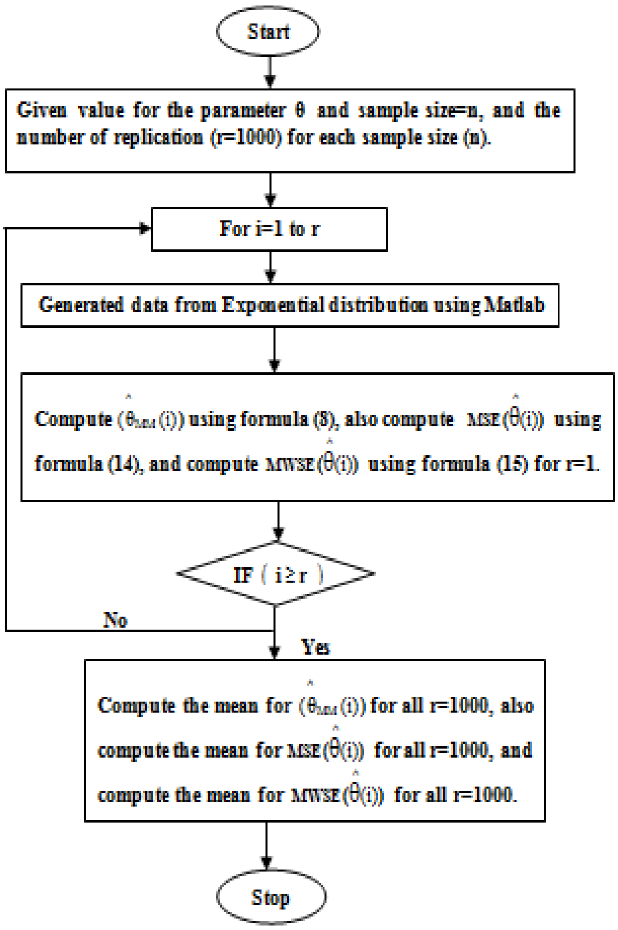
$$\hat{\theta}_{WSE6}=\frac{\Gamma(n+c-1)(\sum_{i=1}^{n}x_i)}{\Gamma(n+c)} \qquad n,c>0 \quad \ldots (C.39)$$

Appendix-D: **The following is the programs algorithm.**

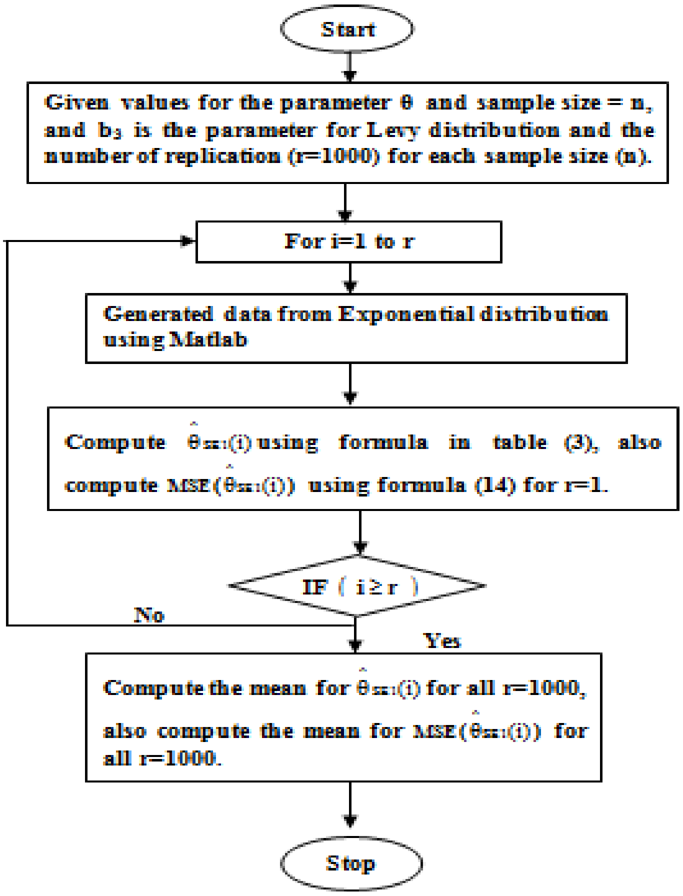
Algorithm (1): To compute MLE for scale parameter ($\hat{\theta}$) with MSE and MWSE.



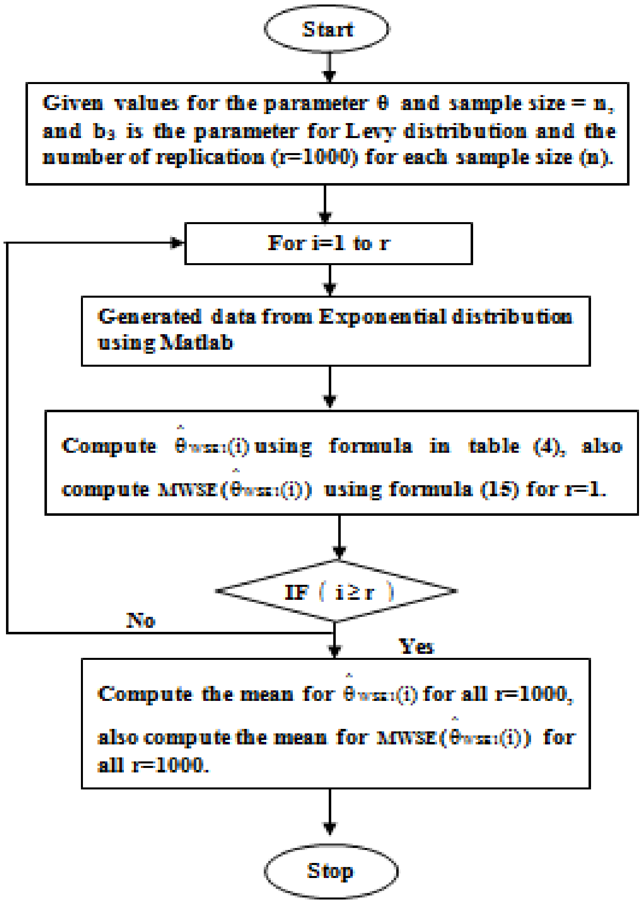
Algorithm (2): To compute MM for scale parameter ($\hat{\theta}$) with MSE and MWSE.



Algorithm (3): To compute Bayes estimators ($\hat{\theta}_{SE1}$) using Levy distribution as prior distribution for θ with MSE.



Algorithm (4): To compute Bayes estimators ($\hat{\theta}_{WSE1}$) using Levy distribution as prior distribution for θ with MWSE.



Note (1): we can reformulate the Algorithm (3) to compute Bayes estimators $\hat{\theta}_{SEk}$, $k = 2,3,4,5,6$ under using other distributions as prior distribution for θ with MSE

Note (2): We can reformulate the Algorithm (4) to compute Bayes estimators $\hat{\theta}_{WSEk}$, $k = 2,3,4,5,6$ under using other distributions as prior distribution for θ with MWSE.

استعمال دوال أولية ودالتين خسارة مختلفة لمقارنة مقدرات بيز مع بعض المقدرات الكلاسيكية لمعلمة التوزيع الاسي

أ.م.د. جنان عباس ناصر العبيدي / الكلية التقنية الادارية / بغداد

المستخلص:

في هذا البحث ، استعملنا طرائق مختلفة لتقدير معلمة القياس للتوزيع الاسي كمقدر الإمكان الأعظم ومقدر العزوم ومقدر بيز في ستة أنواع مختلفة عندما يكون التوزيع الأولي لمعلمة القياس : توزيع لافي (Levy) وتوزيع كامبل من النوع الثاني وتوزيع معكوس مربع كاي وتوزيع معكوس كاما وتوزيع غير الملائم (Improper) وتوزيع Non-informative. وفقا لدالتي الخسارة هي : دالة الخسارة التربيعية و دالة الخسارة التربيعية الموزونة. استعمل أسلوب المحاكاة في مقارنة اداء كل مقدر، بافتراض عدة حالات لمعلمة التوزيع الاسي استعملت لتوليد البيانات ولأحجام مختلفة من العينات (صغيرة ، متوسطة ، كبيرة). وقد أظهرت نتائج المحاكاة بان طريقة بيز الأفضل وفقا لمقياس اقل قيمة متوسط مربع الأخطاء (MSE) ، متوسط مربع الأخطاء الموزونة (MWSE) مقارنة بطريقتي الإمكان الأعظم (MLE) وطريقة العزوم (ME). وفقا للنتائج المستحصلة ، نرى بانه عندما يكون التوزيع الاولي لـ θ توزيع معكوس كاما عند قيم معينة لمعلمتي التوزيع الاولي ، أعطى نتائج أفضل وفقا لاقل قيمة لـ MSE ولـ MWSE مقارنة بنفس القيم المستحصلة بطريقتي MLE وME ،عندما تكون القيمة الحقيقة المفترضة لـ $\theta = 0.5$ ولكل حجوم العينات (n). وعندما يكون التوزيع الاولي لـ θ هو غير الملائم (Improper) عند قيم معينة لمعلمتي التوزيع الاولي، اعطى نتائج أفضل وفقا لاقل قيمة لـ MSE ولـ MWSE مقارنة بنفس القيم المستحصلة بطريقتي MLE وME ، للقيم الحقيقة المفترضة لـ $\theta = 1,2,4$ ولكل حجوم العينات (n) .

المصطلحات الرئيسة للبحث/ التوزيع الآسي ، طريقة الإمكان الأعظم، طريقة العزوم ، طريقة بيز ، متوسط مربع الأخطاء (MSE) ، متوسط مربع الأخطاء الموزونة (MWSE).