A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

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Abstract

A comparison of double informative and non- informative priors assumed for the parameter of Rayleigh distribution is considered. Three different sets of double priors are included, for a single unknown parameter of Rayleigh distribution. We have assumed three double priors: the square root inverted gamma (SRIG) - the natural conjugate family of priors distribution, the square root inverted gamma – the non-informative distribution, and the natural conjugate family of priors - the non-informative distribution as double priors .The data is generating form three cases from Rayleigh distribution for different samples sizes (small, medium, and large). And Bayes estimators for the parameter is derived under a squared error loss function and weighted squared error loss function) in the cases of the three different sets of prior distributions .Simulations is employed to obtain results. And determine the best estimator according to the smallest value of mean squared error and weighted mean squared error. We found that the best estimation for the parameter (θ) for all sample sizes (n) , when the double prior distribution for θ is SRIG - the natural conjugate family of priors distribution with values (a=5, b=0.5, α =8, β =0.5) and (a=8, b=1, α =5, β =1) for the true value of θ ($\theta = 0.5$ and 1) respectively .Also ,we obtained the best estimation for θ when the double prior distribution for θ is the natural conjugate family of priors-non-informative distribution with values(α =0.5, β =5, c=1) for the true value of θ ($\theta = 1.5$).

Key words/ **Rayleigh distribution, Bayes method double informative and non-informative priors, the posterior distribution, the squared error loss function, the weighted squared error loss function.**

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Introduction

The Rayleigh distribution is a member of continuous probability distributions, and it's a suitable model for life testing experiments and clinical studies. We mention some of studies in a brief manner: Hendi et al and others [6] (2007) used Bayes estimation .They obtained bayes estimates for the unknown parameter of the Rayleigh distribution based on upper record values. They derived Bayes' estimators by assuming the quasi-density prior as the prior distribution for the parameter .And he derived Bayes' estimators under two different loss functions which are squared error loss function and linex loss function. Dey [4] (2009) used Bayes estimation to obtained the estimators for the parameter and reliability function of Rayleigh distribution .He derived Bayes' estimators by assuming the natural conjugate family of priors as the prior distribution for the parameter .And he derived Bayes' estimators under two different loss functions which are squared error loss function and linex loss function. Al Mayali and Al-Shaibani [2] (2013) used different estimation methods to estimate the unknown parameter of Rayleigh distribution, which is maximum likelihood estimation, Bayes estimation, shrinkage estimation, and Bayesian shrinkage estimation. They used simulation to obtain the results and compared between these results to find best method.

Pak et al and others [8] (2014) proposed different procedures for estimating the parameter of Rayleigh distribution under progressive type-II censoring when the available observations are described by means of fuzzy information .They conducted simulation study to assess the performance of these procedures. Dey et al and others [5] (2015) used Bayes shrinkage estimation to obtain the estimators for the parameter of Rayleigh distribution; they derived Bayes' estimators by assuming the natural conjugate family of priors as the prior distribution for the parameter of Rayleigh distribution. They derived Bayes' estimators for the parameter of Rayleigh distribution under the assumption of general entropy loss function for progressive Type-II censored data.

The aim of research

The Aim of this paper is to obtain bayes Estimators for the parameter of the Rayleigh distribution under different double informative and non- informative priors. A few studies present in double informative priors, we mention it: Haq and Aslam [1], they used double prior selection for discrete case in the case of Poisson distribution. Radha and Vekatesan (2015), they studied double prior selection for continuous case in the case of Maxwell distribution [9] .They have assumed generalized uniform-inverted Gamma distribution as double priors.

Here we study double prior selection for continuous case in the case of the Rayleigh distribution. We have assumed the square root inverted gamma (SRIG) - the natural conjugate family of priors distribution, the square root inverted gamma – the non-informative distribution, and the natural conjugate family of priors - the non-informative distribution as double priors.

We try to find best method to estimate parameter of Rayleigh distribution. According to the smallest value of Mean Square Errors (MSE) and Mean Weighted Square Errors (MWSE) were calculated to compare the methods of estimation. Several cases from Rayleigh distribution for data generating, of different samples sizes (small, medium, and large).The results were obtained by using simulation technique.

1. The Rayleigh Distribution

Let us consider $t_1, t_2, \dots, t_n$ is a random sample of n independent observations from a Rayleigh distribution having the probability density function (pdf) defined [3]:

$$f(t;\theta) = \frac{t}{\theta^2}\exp\left(-\frac{t^2}{2\theta^2}\right) \quad , \quad t \ge 0, \theta > 0 \qquad \dots (1)$$

where θ is a scale parameter. And the cumulative distribution function (cdf) is given as;

$$F(t;\theta) = 1 - \exp\left(-\frac{t^2}{2\theta^2}\right) \quad , \quad t \ge 0, \theta > 0 \qquad \dots (2)$$

Also, the Reliability function is

$$R(t) = 1 - F(t) = \int_t^\infty f(u)du = \exp\left(-\frac{t^2}{2\theta^2}\right) \quad , \quad t \ge 0, \theta > 0 \qquad \dots (3)$$

Where $R(t)$ is probability of surviving at least till age t, and the failure rate function (or hazard function) at mission time t is given by

$$h(t) = \frac{t}{\theta^2} \quad , \quad t \ge 0, \theta > 0 \qquad \dots (4)$$

3. Bayes Estimation Method

In this section, we used bayes estimation to derived Bayes' estimators for the parameter of Rayleigh distribution, under different double informative priors. Let $t_1, t_2, \dots, t_n$ be a random sample of size n with probability density function given in equation (1) and likelihood function given by equation (5) [3]:

$$L(\underline{t} \backslash \theta) = \prod_{i=1}^{n} f(t;\theta) = \frac{\prod_{i=1}^{n} t_i}{\theta^{2n}} \exp\left(-\frac{\sum_{i=1}^{n} t_i^2}{2\theta^2}\right) \qquad \dots (5)$$

In this paper the posterior distributions for the unknown parameter θ are derived using the following three types of double informative priors, and then get bayes estimation.

Here we have assumed the square root inverted gamma [7]-the natural conjugate family of priors distribution, the square root inverted gamma – the non-informative distribution, and the natural conjugate family of priors - the non-informative distribution as double priors, to get bayes estimation for the parameter of Rayleigh distribution.

3.1 The posterior distribution using different double priors

In this section, we derive the posterior distributions .It is assumed that θ follows three types of prior distributions with pdf as given in table -1:

Table -1: The three types of prior distributions ($f_i(\theta)$) with pdf for θ.

Prior distribution	$f_i(\theta)\quad ,i=1,2,3$
θ ~ SRIG distribution (a,b)	$f_1(\theta)=\frac{2b^a}{\Gamma a}\ \theta^{-(2a+1)}\ \exp(-\frac{b}{\theta^2})$ for $a,b,\ \theta>0$
θ ~ natural conjugate family of priors distribution ($\alpha,\ \beta$)	$f_2(\theta)=\frac{1}{\theta^{\alpha+1}}\ \exp[-\frac{\beta}{2\theta^2}]$ for $\theta,\alpha,\ \beta>0$
θ ~ non-informative distribution (c)	$f_3(\theta)=\frac{1}{\theta^c}$ for $\theta,c>0$

And their double prior's distributions with pdf as given in table -2:

Table -2: The three types of double prior distributions ($P_i(\theta)$) with pdf for θ.

Double prior dist^n.	$P_i(\theta)\quad ,i=1,2,3$
θ ~ SRIG (a,b)- θ ~ the natural conjugate family of priors ($\alpha,\ \beta$)	$P_1(\theta)\ \alpha\ \ f_1(\theta)\ \ f_2(\theta)$ $P_1(\theta)\ \alpha[\frac{2b^a}{\Gamma a}]\theta^{-(2a+\alpha+2)}\exp[-\frac{1}{\theta^2}(b+\beta)]$ for $a,b,\alpha,\ \beta,\theta>0$
θ ~ SRIG (a,b)- θ ~ non-informative distribution (c)	$P_2(\theta)\ \alpha\ \ f_1(\theta)\ \ f_3(\theta)$ $P_2(\theta)\ \alpha[\frac{2b^a}{\Gamma a}]\theta^{-(2a+c+1)}\ \exp[-\frac{b}{\theta^2}]$ for $a,b,c,\theta>0$

$\theta \sim$ **the natural conjugate family of priors** (α, β)**-** $\theta \sim$ **non-informative distribution (c)**	$P_3(\theta)\ \alpha\ \ f_2(\theta)\ \ f_3(\theta)$ $P_3(\theta)\ \ \alpha\,\theta^{-(\alpha+c+1)}\exp[-\frac{\beta}{2\theta^2}]$ for $\alpha, \beta, c, \theta > 0$

Then the posterior distribution of θ **for the given the data** $\underline{t} = (t_1, t_2, \ldots, t_n)$ **is given by:**

$$P(\theta \backslash t) = \frac{L(\underline{t} \backslash \theta)\ \ P(\theta)}{\int_{\theta} L(\underline{t} \backslash \theta)\ \ P(\theta) d\theta} \qquad \ldots(6)$$

Substituting the equation (5) and for each $P(\theta)$ **as shown in table -2 in equation (6), we get the posterior distributions for the unknown parameter θ are derived using the following three types of double priors (for more details see Appendix-A).**

Table -3: The posterior distributions ($P(\theta \backslash t)$ **) for the unknown parameter (θ) are derived using the following three types of double prior distributions.**

Double prior dist^n.	**The posterior distribution (** $P(\theta \backslash t)$ **)**
$\theta \sim$ **SRIG (a,b)-** $\theta \sim$ **the natural conjugate family of priors** (α, β)	$P_1(\theta \backslash t)) \sim$ SRIG dist^n.with ($a_{(new)} = (a + n + 0.5\alpha + 0.5), b_{(new)} = (0.5\sum_{i=1}^{n} t_i^2 + b + \beta)$) $P_1(\theta \backslash t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a + n + 0.5\alpha + 0.5)} \theta^{-(2(a+n+0.5\alpha+0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]$ $a, b, \alpha, \beta, n, \theta \geq 0$
$\theta \sim$ **SRIG (a,b)-** $\theta \sim$ **non-nformative distribution (c)**	$P_2(\theta \backslash t)) \sim$ SRIG dist^n.with ($a_{(new)} = (a + n + 0.5c), b_{(new)} = (0.5\sum_{i=1}^{n} t_i^2 + b)$) $P_2(\theta \backslash t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c)}}{\Gamma(a + n + 0.5c)} \theta^{-(2(a+n+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b)]$ $a, b, c, n, \theta \geq 0$
$\theta \sim$ **the natural conjugate family of priors** (α, β)**-** $\theta \sim$ **non-informative distribution (c)**	$P_3(\theta \backslash t)) \sim$ SRIG dist^n.with ($a_{(new)} = (n + 0.5\alpha + 0.5c), b_{(new)} = (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)$) $P_3(\theta \backslash t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n + 0.5\alpha + 0.5c)} \theta^{-(2(n+0.5\alpha+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]$ $\alpha, \beta, c, n, \theta \geq 0$

3.2 Bayes' Estimators

The objective of this section, to find Bayesian estimators of the scale parameter of Rayleigh distribution under two loss functions and three different of double prior distributions :

- **The squared error loss function** $L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$ **.**
- **The weighted squared error loss function** $L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta}$ **.**

Where $\hat{\theta}$ **is an estimator for** θ **was considered with three different of double prior distributions, and under two loss functions. Following is the derivation of these estimators:**

First: The squared error loss function

To find Bayesian estimators of the scale parameter of Rayleigh distribution under the squared error loss function and three different of double prior distributions:

$$L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \qquad \dots (7)$$

After simplified steps, we get Bayes estimator of θ **denoted by** $\hat{\theta}_{SE}$ **for the above prior as follows**

$$\hat{\theta}_{SE} = E(\theta \backslash t) = \int_0^\infty \theta\, P(\theta \backslash t)\, d\theta \qquad \dots (8)$$

So, the following results are the derivations of these estimators under the squared error loss function with three different double priors (for more details see Appendix-B).

Table -4: The estimators ($\hat{\theta}_{SE}$) under the squared error loss function With three different double priors.

Double Prior distribution	$\hat{\theta}_{SE} = E(\theta \backslash t) = \int_0^\infty \theta P(\theta \backslash t) d\theta$
$\theta\sim$ SRIG (a,b)- $\theta\sim$ the natural conjugate family of priors (α , β)	$\hat{\theta}_{SE1} = \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)} (0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{0.5}$, $n, \alpha, \beta, b, a > 0$
$\theta\sim$ SRIG (a,b)- $\theta\sim$ non-informative distribution (c)	$\hat{\theta}_{SE2} = \frac{\Gamma(a+n+0.5c-0.5)}{\Gamma(a+n+0.5c)} (0.5\sum_{i=1}^{n} t_i^2 + b)^{0.5}$, $n, c, b, a > 0$
$\theta\sim$ the natural conjugate family of priors (α , β)- $\theta\sim$ non-informative distribution (c)	$\hat{\theta}_{SE3} = \frac{\Gamma(n+0.5\alpha+0.5c-0.5)}{\Gamma(n+0.5\alpha+0.5c)} (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{0.5}$, $\alpha, \beta, c, n > 0$

Where $\Gamma(.)$ is a gamma function.

Second: The weighted squared error loss function

To find Bayesian estimators of the scale parameter of Rayleigh distribution under the weighted squared error loss function and three different of double prior distributions:

$$L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta} \qquad \dots (9)$$

After simplified steps, we get Bayes estimator of θ denoted by $\hat{\theta}_{WSE}$ for the above prior as follows

$$\hat{\theta}_{WSE} = \frac{1}{E(\frac{1}{\theta} \backslash t)} = \frac{1}{\int_0^\infty \frac{1}{\theta} P(\theta \backslash t) d\theta} \qquad \dots (10)$$

So, the following results are the derivations of these estimators under the weighted squared error loss function with three different double priors (for more details see Appendix-C).

Table -5: The estimators ($\hat{\theta}_{WSE}$) under the weighted squared error loss with three different double priors.

Double Prior distribution	$\hat{\theta}_{WSE} = \frac{1}{E(\frac{1}{\theta} \setminus t)} = \frac{1}{\int_0^{\infty} \frac{1}{\theta} P(\theta \setminus t)\, d\theta}$
θ ~ **SRIG (a,b)-** θ ~ **the natural conjugate family of priors (** α, β **)**	$\hat{\theta}_{WSE1} = \frac{\Gamma(a+n+0.5\alpha+0.5)\,(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{0.5}}{\Gamma(a+n+0.5\alpha+1)} \quad n, \alpha, \beta, b, a > 0$
θ ~ **SRIG (a,b)-** θ ~ **non-informative distribution(c)**	$\hat{\theta}_{WSE2} = \frac{\Gamma(a+n+0.5c)\ \ (0.5\sum_{i=1}^{n} t_i^2 + b)^{0.5}}{\Gamma(a+n+0.5c+0.5)} \quad n, c, b, a > 0$
θ ~ **the natural conjugate family of priors (** α, β **)-** θ ~ **non-informative distribution (c)**	$\hat{\theta}_{WSE3} = \frac{\Gamma(n+0.5\alpha+0.5c)}{\Gamma(n+0.5\alpha+0.5c+0.5)} (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{0.5}, \alpha, \beta, c, n > 0$

4. Simulation Study

In this study, we have generated random samples from Rayleigh distribution and compared the performance of Bayes estimators based on them. So we have considered several steps to perform simulation study as follows:

1.**We have chosen sample size n = 15, 25, 50 and 100 to represent small, moderate and large sample size.**

2.**We generated data from Rayleigh distribution for the scale parameter; we have considered randomly several values for the parameter of Rayleigh distribution** $\theta = 0.5,\ 1,\ 1.5$ **.**

3.**We used randomly the values for the parameters of the square root inverted gamma (SRIG) distribution with (a , b) - the natural conjugate family of priors distribution with (** α, β **) as double prior distribution for** θ**, when (a , b ,** α, β **) = (5,0.5, 8,0.5) ,(8,0.5, 5,0.5),(5,1,8,1) and (8,1,5,1) .**

4.**We used randomly the values for the parameters of the square root inverted gamma (SRIG) distribution with (a , b) – the non-informative distribution with (c) as double prior distribution for** θ **,when (a , b ,c) = (5, 0.5,2) ,(5, 0.5,3),(5, 0.5,5),(5, 1,2) ,(5, 1,3) and (5, 1,5).**

5.**We used randomly the values for the parameters of the natural conjugate family of priors distribution with (** α, β **) - the non-informative distribution with (c) as double prior distribution for** θ**, when (** α, β**, c) = (0.5, 5, 0.5), (0.5, 5, 1), (0.5, 8, 0.5) and (0.5, 8, 1).**

6.**The number of replication used was (** $r = 1000$ **) for each sample size (n).**

We obtained estimators for scale parameter in table -4, it means the estimators $(\hat{\theta}_{SE})$ **under the squared error loss function with three different double priors .And the estimators in table -5, it means the estimators** $(\hat{\theta}_{WSE})$ **under the weighted squared error loss function with three different double.**

The simulation program was written by using MATLAB-R2008a program. After the parameter θ **was estimated, Mean Square Errors (MSE) and Mean weighted squared Errors (MWSE) were calculated to compare the methods of estimation, where:**

$$MSE = \frac{1}{r}\sum_{r=1}^{1000} (\hat{\theta}(r) - \theta)^2 \qquad ...(11)$$

$$MWSE = \frac{1}{r}\sum_{r=1}^{1000} [(\hat{\theta}(r) - \theta)^2 / \theta] \qquad ...(12)$$

See appendix-D, for the programs algorithm .The results of the simulation study are summarized and tabulated in tables (4-1 to 4-3).In each row of tables (4-1) – (4-3) ,we have four estimated values for θ $(\hat{\theta})$ **with MSE for all samples sizes (n) and values ((a , b), (** α, β **),c) respectively. Also the results of the simulation study are summarized and tabulated in tables (4-4to4-6).In each row of tables (4-4)-(4-6) ,we have four estimated values for** θ $(\hat{\theta})$ **with MWSE for all sample size (n) and values ((a , b), (** α, β **),c) respectively .The Bayes estimators under three types of double prior distribution .So our criteria is the best estimator that gives the smallest value of (MSE) and (MWSE). We list the results in the following tables (4-1 to4-6).**

Table 4-1: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

Method	parameters					Estimate for θ $(\hat{\theta})$				$MSE(\hat{\theta})$			
						Sample Size(n)				Sample Size(n)			
						15	25	50	100	15	25	50	100
Bayes	θ	a	b	α	β	$(P_1(\theta \backslash t))$ when double prior is (SRIG - the natural conjugate family of priors) dist^n.							
	0.5	5	0.5	8	0.5	0.49521	0.49756	0.49867	0.49826	0.00276	0.00188	0.00103	0.00056
		8	0.5	5	0.5	0.51571	0.51083	0.50571	0.50189	0.00321	0.00210	0.00109	0.00057
		5	1	8	1	0.54557	0.53102	0.51689	0.50782	0.00433	0.00261	0.00124	0.00060
		8	1	5	1	0.56816	0.54518	0.52419	0.51151	0.00709	0.00377	0.00157	0.00068
Bayes	θ	a	b	c	-	$(P_2(\theta \backslash t))$ when double prior is (SRIG - the non-informative)) dist^n.							
	0.5	5	0.5	2		0.51728	0.5115	0.50589	0.50191	0.00404	0.00243	0.00117	0.00059
		5	0.5	3		0.50926	0.5066	0.50342	0.50067	0.00371	0.00230	0.00114	0.00059
		5	0.5	5		0.49428	0.49722	0.49858	0.49822	0.00345	0.00218	0.00111	0.00058
		5	1	2		0.54746	0.53029	0.51558	0.50684	0.00559	0.00305	0.00134	0.00062
		5	1	3		0.53898	0.52522	0.51306	0.50559	0.00476	0.00273	0.00125	0.00060
		5	1	5		0.52312	0.51549	0.50813	0.50311	0.00358	0.00225	0.00113	0.00058
Bayes	θ	α	β	c	-	$(P_3(\theta \backslash t))$ when double prior is(the natural conjugate family of priors- non-informative) dist^n.							
	0.5	0.5	5	0.5		0.65005	0.59461	0.54903	0.52396	0.02523	0.01078	0.00341	0.00112
		0.5	5	1		0.64461	0.59163	0.54765	0.52331	0.02358	0.01021	0.00327	0.00109
		0.5	8	0.5		0.72441	0.64377	0.57591	0.53815	0.05255	0.02223	0.00667	0.00198
		0.5	8	1		0.71835	0.64054	0.57447	0.53747	0.04983	0.02130	0.00645	0.00192

Table 4-2: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

Method	parameters					Estimate for θ $(\hat{\theta})$				$MSE(\hat{\theta})$			
						Sample Size(n)				Sample Size(n)			
						15	25	50	100	15	25	50	100
Bayes	θ	a	b	α	β	$(P_1(\theta \backslash t))$ when double prior is (SRIG - the natural conjugate family of priors) dist^n.							
	1	5	0.5	8	0.5	0.89565	0.93375	0.96466	0.9796	0.02359	0.01267	0.00556	0.00271
		8	0.5	5	0.5	0.93172	0.95822	0.97816	0.98669	0.01841	0.01046	0.00491	0.00251
		5	1	8	1	0.92389	0.95175	0.97413	0.98447	0.01773	0.01029	0.00490	0.00252
		8	1	5	1	0.9611	0.97668	0.98775	0.9916	0.01443	0.00893	0.00450	0.00238
Bayes	θ	a	b	c	-	$(P_2(\theta \backslash t))$ when double prior is (SRIG - the non-informative) dist^n.							
	1	5	0.5	2		0.97141	0.98432	0.99213	0.9939	0.01676	0.00982	0.00472	0.00243
		5	0.5	3		0.95681	0.97508	0.98733	0.99146	0.01734	0.01001	0.00477	0.00245
		5	0.5	5		0.92947	0.95735	0.97792	0.98662	0.01957	0.01087	0.00501	0.00253
		5	1	2		0.98738	0.99405	0.99706	0.99639	0.01559	0.00942	0.00462	0.00239
		5	1	3		0.97254	0.98471	0.99223	0.99394	0.01573	0.00944	0.00463	0.00240
		5	1	5		0.94475	0.96681	0.98278	0.98909	0.01718	0.00997	0.00478	0.00246
Bayes	θ	α	β	c	-	$(P_3(\theta \backslash t))$ when double prior is(the natural conjugate family of priors- non-informative) dist^n.							
	1	0.5	5	0.5		1.085	1.0532	1.0268	1.0113	0.02274	0.01222	0.00533	0.00251
		0.5	5	1		1.0759	1.0479	1.0243	1.0101	0.02103	0.01159	0.00518	0.00247
		0.5	8	0.5		1.1314	1.0818	1.0415	1.0188	0.03156	0.01558	0.00620	0.00269
		0.5	8	1		1.122	1.0764	1.0389	1.0175	0.02892	0.01463	0.00597	0.00264

Table 4-3: Shows the values for $\hat{\theta}$ under square error loss function (MSE).

Method	parameters					Estimate for θ $(\hat{\theta})$				$MSE(\hat{\theta})$			
						Sample Size(n)				Sample Size(n)			
						15	25	50	100	15	25	50	100
Bayes	θ	a	b	α	β	$(P_1(\theta \backslash t))$ when double prior is (SRIG - the natural conjugate family of priors) dist^n.							
	1.5	5	0.5	8	0.5	1.3029	1.3739	1.4325	1.4618	0.06775	0.03464	0.01429	0.00664
		8	0.5	5	0.5	1.354	1.4093	1.4524	1.4724	0.05253	0.02794	0.01227	0.00602
		5	1	8	1	1.322	1.386	1.4389	1.4651	0.05974	0.03140	0.01339	0.00637
		8	1	5	1	1.3739	1.4217	1.4588	1.4756	0.04622	0.0255	0.01162	0.00582
Bayes	θ	a	b	c	-	$(P_2(\theta \backslash t))$ when double prior is (SRIG - the non-informative)) dist^n.							
	1.5	5	0.5	2		1.4219	1.4545	1.4769	1.4851	0.04157	0.02345	0.01097	0.00559
		5	0.5	3		1.4012	1.4411	1.4698	1.4815	0.04421	0.02446	0.01125	0.00569
		5	0.5	5		1.3622	1.4154	1.4559	1.4743	0.05153	0.02741	0.01209	0.00595
		5	1	2		1.4326	1.461	1.4802	1.4868	0.03949	0.02271	0.01079	0.00553
		5	1	3		1.4117	1.4475	1.4731	1.4831	0.04174	0.02355	0.01102	0.00562
		5	1	5		1.3724	1.4217	1.4592	1.4759	0.04834	0.02619	0.01177	0.00586
Bayes	θ	α	β	c	-	$(P_3(\theta \backslash t))$ when double prior is(the natural conjugate family of priors- non-informative) dist^n.							
	1.5	0.5	5	0.5		1.5601	1.5389	1.5196	1.5066	0.04161	0.02381	0.01105	0.00547
		0.5	5	1		1.547	1.5312	1.5158	1.5047	0.03957	0.02304	0.01087	0.00544
		0.5	8	0.5		1.5928	1.5587	1.5296	1.5116	0.04507	0.02516	0.01140	0.00553
		0.5	8	1		1.5795	1.5508	1.5257	1.5097	0.04217	0.02409	0.01114	0.00547

Table 4-4: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

Method	parameters					Estimate for θ $(\hat{\theta})$				$MWSE(\hat{\theta})$			
						Sample Size(n)				Sample Size(n)			
						15	25	50	100	15	25	50	100
Bayes	θ	a	b	α	β	$(P_1(\theta \backslash t))$ when double prior is (SRIG - the natural conjugate family of priors) dist^n.							
	0.5	5	0.5	8	0.5	0.4889	0.49336	0.49639	0.49707	0.00558	0.00378	0.00206	0.00113
		8	0.5	5	0.5	0.50859	0.50629	0.50333	0.50067	0.00592	0.00397	0.00211	0.00113
		5	1	8	1	0.53863	0.52654	0.51453	0.50661	0.00739	0.00465	0.00231	0.00116
		8	1	5	1	0.56032	0.54034	0.52172	0.51027	0.01204	0.00666	0.00289	0.00130
Bayes	θ	a	b	c	-	$(P_2(\theta \backslash t))$ when double prior is (SRIG - the non-informative) dist^n.							
	0.5	5	0.5	2		0.50926	0.5066	0.50342	0.50067	0.00742	0.00460	0.00228	0.00118
		5	0.5	3		0.5016	0.50185	0.50098	0.49944	0.00704	0.00443	0.00224	0.00117
		5	0.5	5		0.48727	0.49272	0.49621	0.497	0.00696	0.00437	0.00222	0.00117
		5	1	2		0.53898	0.52522	0.51306	0.50559	0.00952	0.00546	0.00251	0.00121
		5	1	3		0.53087	0.52029	0.51057	0.50435	0.00819	0.00493	0.00238	0.00118
		5	1	5		0.51571	0.51083	0.50571	0.50189	0.00643	0.00420	0.00218	0.00114
Bayes	θ	α	β	c	-	$(P_3(\theta \backslash t))$ when double prior is(the natural conjugate family of priors- non-informative) dist^n.							
	0.5	0.5	5	0.5		0.63931	0.58869	0.54629	0.52265	0.04407	0.01933	0.00628	0.00212
		0.5	5	1		0.63413	0.5858	0.54493	0.522	0.04115	0.01828	0.00602	0.00206
		0.5	8	0.5		0.71244	0.63736	0.57304	0.5368	0.09450	0.04080	0.01248	0.00375
		0.5	8	1		0.70667	0.63423	0.57162	0.53614	0.08960	0.03907	0.01206	0.00365

Table 4-5: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

Method	parameters					Estimate for θ $(\hat{\theta})$				MWSE$(\hat{\theta})$			
						Sample Size(n)				Sample Size(n)			
						15	25	50	100	15	25	50	100
Bayes	θ	a	b	α	β	$(P_1(\theta \backslash t))$ when double prior is (SRIG - the natural conjugate family of priors) dist^n.							
	1	5	0.5	8	0.5	0.88452	0.926	0.96029	0.97727	0.02572	0.01362	0.00585	0.00280
		8	0.5	5	0.5	0.91922	0.94985	0.9736	0.98431	0.01990	0.01108	0.00509	0.00257
		5	1	8	1	0.91241	0.94385	0.96971	0.98213	0.01931	0.01098	0.00511	0.00258
		8	1	5	1	0.9482	0.96815	0.98315	0.98921	0.01526	0.00925	0.00459	0.00241
Bayes	θ	a	b	c	-	$(P_2(\theta \backslash t))$ when double prior is (SRIG - the non-informative) dist^n.							
	1	5	0.5	2		0.95681	0.97508	0.98733	0.99146	0.01734	0.01001	0.00477	0.00245
		5	0.5	3		0.94284	0.9661	0.98259	0.98903	0.01829	0.01037	0.00487	0.00249
		5	0.5	5		0.91665	0.94884	0.97332	0.98423	0.02115	0.01151	0.00519	0.00259
		5	1	2		0.97254	0.98471	0.99223	0.99394	0.01573	0.00944	0.00463	0.00240
		5	1	3		0.95834	0.97564	0.98747	0.9915	0.01628	0.00963	0.00468	0.00243
		5	1	5		0.93172	0.95822	0.97816	0.98669	0.01841	0.01046	0.00491	0.00251
Bayes	θ	α	β	c	-	$(P_3(\theta \backslash t))$ when double prior is(the natural conjugate family of priors- non-informative) dist^n.							
	1	0.5	5	0.5		1.0671	1.0427	1.0217	1.0088	0.01951	0.01103	0.00504	0.00244
		0.5	5	1		1.0584	1.0376	1.0192	1.0076	0.01818	0.01052	0.00491	0.00242
		0.5	8	0.5		1.1127	1.071	1.0363	1.0162	0.02652	0.01376	0.00575	0.00259
		0.5	8	1		1.1037	1.0658	1.0337	1.0149	0.02435	0.01295	0.00555	0.00255

Table 4-6: Shows the values for $\hat{\theta}$ under weighted square error loss function (MWSE).

Method	parameters					Estimate for θ $(\hat{\theta})$				MWSE$(\hat{\theta})$			
						Sample Size(n)				Sample Size(n)			
						15	25	50	100	15	25	50	100
Bayes	θ	a	b	α	β	$(P_1(\theta \backslash t))$ when double prior is (SRIG - the natural conjugate family of priors) dist^n.							
	1.5	5	0.5	8	0.5	1.2871	1.3627	1.4261	1.4584	0.04901	0.02486	0.01007	0.00459
		8	0.5	5	0.5	1.3363	1.3972	1.4457	1.4688	0.03813	0.01996	0.00858	0.00413
		5	1	8	1	1.306	1.3747	1.4324	1.4616	0.04335	0.02254	0.00942	0.00440
		8	1	5	1	1.3559	1.4095	1.4521	1.4721	0.03352	0.01815	0.00808	0.00399
Bayes	θ	a	b	c	-	$(P_2(\theta \backslash t))$ when double prior is (SRIG - the non-informative) dist^n.							
	1.5	5	0.5	2		1.4012	1.4411	1.4698	1.4815	0.02947	0.01630	0.00750	0.00379
		5	0.5	3		1.3813	1.4281	1.4628	1.4779	0.03171	0.01719	0.00775	0.00387
		5	0.5	5		1.344	1.403	1.4491	1.4707	0.03735	0.01953	0.00842	0.00408
		5	1	2		1.4117	1.4475	1.4731	1.4831	0.02782	0.01570	0.00734	0.00374
		5	1	3		1.3916	1.4344	1.4661	1.4795	0.02981	0.01648	0.00756	0.00382
		5	1	5		1.354	1.4093	1.4524	1.4724	0.03502	0.01863	0.00818	0.00401
Bayes	θ	α	β	c	-	$(P_3(\theta \backslash t))$ when double prior is(the natural conjugate family of priors- non-informative) dist^n.							
	1.5	0.5	5	0.5		1.5343	1.5236	1.512	1.5029	0.02528	0.01494	0.00714	0.00360
		0.5	5	1		1.5219	1.5161	1.5083	1.501	0.02442	0.01460	0.00705	0.00359
		0.5	8	0.5		1.5665	1.5431	1.5219	1.5078	0.02645	0.01543	0.00727	0.00362
		0.5	8	1		1.5538	1.5356	1.5182	1.506	0.02506	0.0149	0.00713	0.00359

5. Discussion

We obtained a good estimation according to the smallest values of MSE for all samples sizes (n) comparative to the other estimated values for MSE as shown in tables (4-1 to 4-3) .

As we see <u>in table (4-1),</u> when the true value of $\theta\,(\theta = 0.5)$ and the double prior distribution for θ is

- **SRIG - the natural conjugate family of priors distribution with (a=5, b=0.5, α=8, β=0.5).**
- **SRIG - the non-informative distribution with (a=5, b=0.5, c =5).**
- **The natural conjugate family of priors-non-informative distribution with (α=0.5, β=5, c=1).**

And <u>in tables (4-2) and (4-3),</u> when the true value of $\theta\,(\theta = 1\&1.5)$ and the double prior distribution for θ is

- **SRIG - the natural conjugate family of priors distribution with (a=8, b=1, α=5, β=1).**
- **SRIG - the non-informative distribution with (a=5, b=1, c =2).**
- **The natural conjugate family of priors-non-informative distribution with (α=0.5, β=5, c=1).**

Also, we obtained a good estimation according to the smallest values of MWSE for all samples sizes (n) comparative to the other estimated values for MWSE as shown in tables(4-4 to 4-6) .

<u>In table (4-4),</u> when the true value of $\theta\,(\theta = 0.5)$ and the double prior distribution for θ is

- **SRIG - the natural conjugate family of priors distribution with (a=5, b=0.5, α=8, β=0.5).**
- **SRIG - the non-informative distribution with (a=5, b=1, c =5).**
- **The natural conjugate family of priors-non-informative distribution with (α=0.5, β=5, c=1).**

And <u>in tables (4-5),</u> when the true value of $\theta\,(\theta = 1)$ and the double prior distribution for θ is

- **SRIG - the natural conjugate family of priors distribution with (a=8, b=1, α=5, β=1).**
- **SRIG - the non-informative distribution with (a=5, b=1, c =2).**
- **The natural conjugate family of priors-non-informative distribution with (α=0.5, β=5, c=1).**

And <u>in tables (4-6),</u> when the true value of $\theta\,(\theta = 1.5)$ and the double prior distribution for θ is

- **SRIG - the natural conjugate family of priors distribution with (a=8, b=1, α=5, β=1).**
- **SRIG - the non-informative distribution with (a=5, b=1, c =2).**

- **The natural conjugate family of priors-non-informative distribution with (α=0.5, β=5, c=1).**

For more details see Appendix-E.

6. Conclusion

When we compared the estimated values for θ ($\hat{\theta}$) the parameter of the Rayleigh distribution by using Bayes with respect to Mean Square Errors (MSE) and Mean weighted squared Errors (MWSE) respect to the smallest values of them .In general , we found best estimation for the parameter (θ) for all sample sizes (n) , when the double prior distribution for θ is SRIG - the natural conjugate family of priors distribution with values $(a=5, b=0.5, \alpha=8, \beta=0.5)$ and $(a=8, b=1, \alpha=5, \beta=1)$ for the true value of θ ($\theta=0.5$ and 1) respectively .Also ,we obtained the best estimation when the double prior distribution for θ is the natural conjugate family of priors-non-informative distribution with values ($\alpha=0.5, \beta=5, c=1$) for the true value of θ ($\theta=1.5$).

References

1.Haq, A. and Aslam M., (By internet),' On double prior selection of Poisson distribution', Quaid-I- Azam University, Islamabad, 45320, Pakistan. https://www. qau. edu. pk/ profile. php.

2.Al Mayali, Y., M., and Al_ Shaibani, I., A., (2013),'A Comparison for Some of the Estimators of Rayleigh Distribution with Simulation', Journal of Kerbala University, Vol. 11 No.4. Scientific.

3.Bickel, P.J. & Doksum, K. A., (1977), Mathematical Statistics: Basic Ideas and Selected Topics. Holden- Day, Inc., San Francisco.

4.Dey ,S., (2009), 'Comparison of Bayes Estimators of the Parameter and Reliability Function for Rayleigh Distribution under Different Loss Functions', Malaysian Journal of Mathematical Sciences, 3(2), 249-266.

5.Dey, S., Dey, T., Maiti, S., (2015), 'Bayes Shrinkage Estimation of the Parameter of Rayleigh Distribution for Progressive Type-II Censored Data', Austrian Journal of Statistic, Dec. 2015, vol. 44, 3-15. http//www. ajs. or. at/. Doi: 10.17713 / ajs.v44i4.71.

6.Hendi, M. I., Abu-Youssef, S. E. And Alraddadi, A. A. (2007),'A Bayesian Analysis of Record Statistics from the Rayleigh Model'. International Mathematical Forum no. 13, 619 – 631.

7.Ohakwe, J. and Okolic, N. and Obij, C. and Ugwud, N., (2012),' Properties of a square root gamma distribution', J. Math. Comput. Sci. 2, No. 6, 1588-1597.ISSN: 1927-5307.

8.Pak, A., Gholam Parham, A., and Saraj, M. (2014),'Inference for the Rayleigh Distribution Based on Progressive Type-II Fuzzy Censored Data'. Journal of Modern Applied Methods, Vol.13, Issue 1, Article 19.

9.Radha, R.K. & Vekatesan, P., (2015), 'On the double prior selection for the parameter of Maxwell distribution', International Journal of Scientific & Engineering Research, Volume 4, Issue 5, May-2013.

Appendix-A: The posterior distribution by using different double Priors.

1- **The posterior distribution using SRIG - natural conjugate family of priors distribution as double prior:**

To find the posterior distribution using the square root inverted gamma distribution (SRIG) - the natural conjugate family of priors distribution we follow these steps:

When the prior distribution of θ take SRIG distribution, then the pdf is given by:

$$f_1(\theta) = \frac{2b^a}{\Gamma a}\,\theta^{-(2a+1)}\exp(-\frac{b}{\theta^2}) \qquad \text{for } a, b, \theta > 0 \qquad \ldots(A.1)$$

Again, when prior distribution of θ take the natural conjugate family of priors distribution, then the p.d.f. is given by:

$$f_2(\theta) = \frac{1}{\theta^{\alpha+1}}\exp[-\frac{\beta}{2\theta^2}] \qquad \text{for } \theta, \alpha, \beta > 0 \qquad \ldots(A.2)$$

We define the double prior for θ by combining these two priors as follows:

$$P_1(\theta) \propto f_1(\theta)\; f_2(\theta) \qquad \ldots(A.3)$$

$$P_1(\theta) \propto [\frac{2b^a}{\Gamma a}\,\theta^{-(2a+1)}\exp(-\frac{b}{\theta^2})\,][\frac{1}{\theta^{\alpha+1}}\exp[-\frac{\beta}{2\theta^2}]\,] \qquad \ldots(A.4)$$

$$P_1(\theta) \propto [\frac{2b^a}{\Gamma a}]\theta^{-(2a+\alpha+2)}\exp[-\frac{1}{\theta^2}(b+\beta)\,] \quad \text{for } a, b, \alpha, \beta, \theta > 0 \qquad \ldots(A.5)$$

Then the posterior distribution of θ for the given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is given by:

$$P(\theta \setminus t) = \frac{L(\underline{t}\setminus\theta)\;P(\theta)}{\int_\theta L(\underline{t}\setminus\theta)\;P(\theta)d\theta} \qquad \ldots(A.6)$$

Substituting the equation (5) and (A.5) in equation (A.6), we get:

$$P_1(\theta \setminus t) = \frac{\frac{\prod_{i=1}^n t_i}{\theta^{2n}}\exp(-\frac{\sum_{i=1}^n t_i^2}{2\theta^2})[\frac{2b^a}{\Gamma a}]\theta^{-(2a+\alpha+2)}\exp[-\frac{1}{\theta^2}(b+\beta)]}{\int_0^\infty \frac{\prod_{i=1}^n t_i}{\theta^{2n}}\exp(-\frac{\sum_{i=1}^n t_i^2}{2\theta^2})[\frac{2b^a}{\Gamma a}]\theta^{-(2a+\alpha+2)}\exp[-\frac{1}{\theta^2}(b+\beta)]d\theta} \qquad \ldots(A.7)$$

$$P_1(\theta \setminus t) = \frac{\theta^{-(2a+2n+\alpha+2)}\exp[-\frac{1}{\theta^2}(\frac{\sum_{i=1}^n t_i^2}{2}+b+\beta)]}{\int_0^\infty \theta^{-(2a+2n+\alpha+2)}\exp[-\frac{1}{\theta^2}(\frac{\sum_{i=1}^n t_i^2}{2}+b+\beta)]\,d\theta} \qquad \ldots(A.8)$$

We can rewrite $(2a+2n+\alpha+2)$ as $(2(a+n+0.5\alpha+0.5)+1)$, then by multiplying the integral in equation (A.8) by the quantity which equals to

$$\left(\frac{2(0.5\sum_{i=1}^n t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)}\right)\left(\frac{\Gamma(a+n+0.5\alpha+0.5)}{2(0.5\sum_{i=1}^n t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)}}\right), \text{ where}$$

$\Gamma(.)$ is a gamma function. Then we get,

$$P_1(\theta \backslash t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)A(t,\theta)} \theta^{-(2(a+n+0.5\alpha+0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)] \quad \ldots (A.9)$$

Where $A(t,\theta)$ equals to

$$A(t,\theta) = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)} \theta^{-(2(a+n+0.5\alpha+0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta = 1$$

Be the integral of the pdf of SRIG distribution. Then we get the posterior distribution of θ given the data of θ given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is

$$P_1(\theta \backslash t) = \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)} \theta^{-(2(a+n+0.5\alpha+0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]$$

$$a, b, \alpha, \beta, n, \theta \geq 0 \quad \ldots (A.10)$$

It means that $P_1(\theta \backslash t)) \sim$ SRIG distribution with new parameters ($a_{(new)} = (a + n + 0.5\alpha + 0.5)$, $b_{(new)} = (0.5\sum_{i=1}^{n} t_i^2 + b + \beta)$).

2- **The posterior distribution using SRIG - non-informative distribution as double prior:**

To find the posterior distribution using SRIG -non-informative distribution, we follow these steps:
When the prior distribution of θ take SRIG distribution, then the pdf is given by:

$$f_1(\theta) = \frac{2b^a}{\Gamma a} \theta^{-(2a+1)} \exp(-\frac{b}{\theta^2}) \qquad \text{for } a, b, \theta > 0 \qquad \ldots (A.1)$$

Again, when prior distribution of θ take non-informative distribution, then the pdf is given by:

$$f_3(\theta) = \frac{1}{\theta^c} \qquad \text{for } \theta, c > 0 \qquad \ldots (A.11)$$

We define the double prior for θ by combining these two priors as follows:

$$P_2(\theta) \;\alpha\; f_1(\theta)\; f_3(\theta) \qquad \ldots (A.12)$$

$$P_2(\theta) \;\alpha [\frac{2b^a}{\Gamma a} \theta^{-(2a+1)} \exp(-\frac{b}{\theta^2})][\frac{1}{\theta^c}] \qquad \ldots (A.13)$$

$$P_2(\theta) \;\alpha [\frac{2b^a}{\Gamma a}] \theta^{-(2a+c+1)} \exp[-\frac{b}{\theta^2}] \qquad \text{for } a, b, c, \theta > 0 \qquad \ldots (A.14)$$

Then the posterior distribution of θ for the given the data $\underline{t} = (t_1, t_2, \ldots, t_n)$ is given by:

$$P(\theta \backslash t) = \frac{L(\underline{t} \backslash \theta)\; P(\theta)}{\int_\theta L(\underline{t} \backslash \theta)\; P(\theta) d\theta} \qquad \ldots (A.6)$$

Substituting the equation (5) and (A.14) in equation (A.6), we get:

$$P_2(\theta \backslash t) = \frac{\frac{\prod_{i=1}^{n} t_i}{\theta^{2n}} \exp(-\frac{\sum_{i=1}^{n} t_i^2}{2\theta^2})[\frac{2b^a}{\Gamma a}]\theta^{-(2a+c+1)} \exp[-\frac{b}{\theta^2}]}{\int_0^\infty \frac{\prod_{i=1}^{n} t_i}{\theta^{2n}} \exp(-\frac{\sum_{i=1}^{n} t_i^2}{2\theta^2})[\frac{2b^a}{\Gamma a}]\theta^{-(2a+c+1)} \exp[-\frac{b}{\theta^2}] d\theta} \qquad \ldots (A.15)$$

$$P_2(\theta \backslash t)=\frac{\theta^{-(2a+2n+c+1)}\exp[-\frac{1}{\theta^2}(\frac{\sum_{i=1}^{n}t_i^2}{2}+b)]}{\int_0^\infty \theta^{-(2a++2n+c+1)}\exp[-\frac{1}{\theta^2}(\frac{\sum_{i=1}^{n}t_i^2}{2}+b)]d\theta} \quad \ldots (A.16)$$

We can rewrite $(2a+2n+c+1)$ as $(2(a+n+0.5c)+1)$, then by multiplying the integral in equation (A.16) by the quantity which equals to

$$\left(\frac{2(0.5\sum_{i=1}^{n}t_i^2+b)^{(a+n+0.5c)}}{\Gamma(a+n+0.5c)}\right)\left(\frac{\Gamma(a+n+0.5c)}{2(0.5\sum_{i=1}^{n}t_i^2+b)^{(a+n+0.5c)}}\right)$$, where $\Gamma(.)$ is a gamma

function .Then we get,

$$P_2(\theta \backslash t)=\frac{2(0.5\sum_{i=1}^{n}t_i^2+b)^{(a+n+0.5c)}}{\Gamma(a+n+0.5c)A_1(t,\theta)}\theta^{-(2(a+n+0.5c)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b)] \quad \ldots (A.17)$$

Where $A_1(t,\theta)$ equals to

$$A_1(t,\theta)=\int_0^\infty \frac{2(0.5\sum_{i=1}^{n}t_i^2+b)^{(a+n+0.5c)}}{\Gamma(a+n+0.5c)}\theta^{-(2(a+n+0.5c)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b)]\,d\theta=1.$$ Be the integral of the pdf of Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t}=(t_1,t_2,\ldots,t_n)$ is

$$P_2(\theta \backslash t)=\frac{2(0.5\sum_{i=1}^{n}t_i^2+b)^{(a+n+0.5c)}}{\Gamma(a+n+0.5c)}\theta^{-(2(a+n+0.5c)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b)]$$

$a, b, c, n, \theta \geq 0 \quad \ldots (A.18)$

It means that $P_2(\theta \backslash t)) \sim$ SRIG distribution with new parameters($a_{(new)}=(a+n+0.5c), b_{(new)}=(0.5\sum_{i=1}^{n}t_i^2+b)$).

3- **<u>The posterior distribution using the natural conjugate family of priors- non-informative distribution as double prior:</u>**

To find the posterior distribution using the natural conjugate family of priors- non-informative distribution, we follow these steps:

When the prior distribution of θ take the natural conjugate family of priors distribution, then the pdf is given by:

$$f_2(\theta)=\frac{1}{\theta^{\alpha+1}}\exp[-\frac{\beta}{2\theta^2}] \qquad \text{for} \quad \theta, \alpha, \beta>0 \qquad \ldots (A.2)$$

Again, when prior distribution of θ take Erlang distribution, then the pdf is given by:

$$f_3(\theta)=\frac{1}{\theta^c} \qquad \text{for} \quad \theta, c>0 \qquad \ldots (A.11)$$

We define the double prior for θ by combining these two priors as follows:

$$P_3(\theta)\ \alpha\ f_2(\theta)\ f_3(\theta) \qquad \ldots (A.19)$$

$$P_3(\theta)\ \alpha[\frac{1}{\theta^{\alpha+1}}\exp[-\frac{\beta}{2\theta^2}]][\frac{1}{\theta^c}] \qquad \ldots (A.20)$$

$$P_3(\theta) \;\alpha\; \theta^{-(\alpha+c+1)} \exp[-\frac{\beta}{2\theta^2}] \qquad \text{for } \alpha, \beta, c, \theta > 0 \qquad \dots (A.21)$$

Then the posterior distribution of θ for the given the data $\underline{t} = (t_1, t_2, \dots, t_n)$ is given by:

$$P(\theta \backslash t) = \frac{L(\underline{t} \backslash \theta)\; P(\theta)}{\int_\theta L(\underline{t} \backslash \theta)\; P(\theta) d\theta} \qquad \dots (A.6)$$

Substituting the equation (5) and (A.21) in equation (A.6), we get:

$$P_3(\theta \backslash t) = \frac{\theta^{-(2n+\alpha+c+1)} \exp[-\frac{1}{\theta^2}(\frac{\sum_{i=1}^n t_i^2}{2} + \frac{\beta}{2})]}{\int_0^\infty \theta^{-(2n+\alpha+c+1)} \exp[-\frac{1}{\theta^2}(\frac{\sum_{i=1}^n t_i^2}{2} + \frac{\beta}{2})] d\theta} \qquad \dots (A.22)$$

We can rewrite $(2n+\alpha+c+1)$ as $(2(n+0.5\alpha+0.5c)+1)$, then by multiplying the integral in equation (A.22) by the quantity which equals to

$$\left(\frac{2(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n+0.5\alpha+0.5c)}\right)\left(\frac{\Gamma(n+0.5\alpha+0.5c))}{2(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}\right)$$

, where $\Gamma(.)$ is a gamma function .Then we get,

$$P_3(\theta \backslash t) = \frac{2(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n+0.5\alpha+0.5c)\; A_3(t,\theta)} \theta^{-(2(n+0.5\alpha+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)] \dots (A.23)$$

Where $A_3(t,\theta)$ equals to

$$A_3(t,\theta) = \int_0^\infty \frac{2(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n+0.5\alpha+0.5c)} \theta^{-(2(n+0.5\alpha+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)]\; d\theta = 1.$$

Be the integral of the pdf of Gamma distribution. Then we get the posterior distribution of θ given the data $\underline{t} = (t_1, t_2, \dots, t_n)$ is

$$P_3(\theta \backslash t) = \frac{2(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n+0.5\alpha+0.5c)} \theta^{-(2(n+0.5\alpha+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^n t_i^2 + 0.5\beta)]$$

$$, \alpha, \beta, c, n, \theta \geq 0 \qquad \dots (A.24)$$

It means that $P_3(\theta \backslash t)) \sim$ SRIG distribution with new parameters($a_{(new)} = (n + 0.5\alpha + 0.5c)$, $b_{(new)} = (0.5\sum_{i=1}^n t_i^2 + 0.5\beta)$) .

The following is the derivation of these estimators under the squared error loss function.

The squared error loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_1(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, the risk function is:

$$R(\hat{\theta} - \theta) = E[L_1(\hat{\theta}, \theta)] \qquad \text{...(B.1)}$$

$$R(\hat{\theta} - \theta) = \int_{\theta} L_1(\hat{\theta}, \theta)\, P(\theta \setminus x)\, d\theta$$

$$R(\hat{\theta} - \theta) = \int_{\theta} (\hat{\theta} - \theta)^2\, P(\theta \setminus t)\, d\theta \quad \Rightarrow R(\hat{\theta} - \theta) = \int_{\theta} (\hat{\theta}^2 - 2\theta\hat{\theta} + \theta^2)\, P(\theta \setminus t)\, d\theta$$

$$R(\hat{\theta} - \theta) = \hat{\theta}^2 \int_0^{\infty} P(\theta \setminus t)\, d\theta - 2\hat{\theta} \int_0^{\infty} \theta P(\theta \setminus t)\, d\theta + \int_0^{\infty} \theta^2 P(\theta \setminus t)\, d\theta \Rightarrow$$

$$R(\hat{\theta} - \theta) = \hat{\theta}^2 - 2\hat{\theta} E(\theta \setminus t) + E(\theta^2 \setminus t) \qquad \text{... (B.2)}$$

Let $\dfrac{\partial}{\partial \hat{\theta}} R(\hat{\theta} - \theta) = 0$, we get Bayes estimator of θ denoted by $\hat{\theta}_{Bayes}$ for the above prior as follows

$$\hat{\theta}_{SE} = E(\theta \setminus x) = \int_0^{\infty} \theta P(\theta \setminus x)\, d\theta \qquad \text{... (B.3)}$$

$$\hat{\theta}_{SE} = \int_0^{\infty} \theta P_i(\theta \setminus x)\, d\theta \quad ; \; i = 1,2,3 \qquad \text{... (B.4)}$$

1. **Bayes estimation using SRIG - natural conjugate family of priors distribution as double prior:**

To obtain the Bayes' estimator under SRIG - natural conjugate family of priors distribution as double prior. Substituting the equation (A.10) in equation (B.4), we get:

$$\hat{\theta}_{SE1} = \int_0^{\infty} \theta P_1(\theta \setminus x)\, d\theta \quad ; \; i = 1 \qquad \text{... (B.4)}$$

$$\hat{\theta}_{SE1} = \int_0^{\infty} \theta \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a + n + 0.5\alpha + 0.5)} \theta^{-(2(a+n+0.5\alpha+0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta \qquad \text{... (B.5)}$$

$$\hat{\theta}_{SE1} = \int_0^{\infty} \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a + n + 0.5\alpha + 0.5)} \theta^{-(2(a+n+0.5\alpha+0.5-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta \qquad \text{... (B.6)}$$

$$\hat{\theta}_{ss1} = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)} \theta^{-(2(a+n+0.5\alpha)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta \qquad \ldots (B.7)$$

By multiplying the integral in equation (B.7) by the quantity which equals to $B1 = (\frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha)})$, where $\Gamma(.)$is a gamma function. Then we have

$$\hat{\theta}_{ss1} = B1\int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)} \theta^{-(2(a+n+0.5\alpha)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta \qquad \ldots (B.8)$$

Then, we have

$$\hat{\theta}_{ss1} = \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{0.5} B2(t,\theta) \qquad \ldots (B.9)$$

Where $B2(t,\theta)$ equals to

$$B2(t,\theta) = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha)}}{\Gamma(a+n+0.5\alpha)} \theta^{-(2(a+n+0.5\alpha)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta = 1$$

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of $\hat{\theta}_{ss1}$ as the following formula:

$$\hat{\theta}_{ss1} = \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{0.5} \qquad n, \alpha, \beta, b, a > 0 \qquad \ldots (B.10)$$

2. **Bayes estimation using SRIG - non-informative distribution as double prior:**

To obtain the Bayes' estimator under SRIG - non-informative_distribution as double prior. Substituting the equation (A.18) in equation (B.4), we get:

$$\hat{\theta}_{ss2} = \int_0^\infty \theta P_i(\theta \setminus x)\, d\theta \quad ; \quad i = 2 \qquad \ldots (B.4)$$

$$\hat{\theta}_{ss2} = \int_0^\infty \theta \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c)}}{\Gamma(a+n+0.5c)} \theta^{-(2(a+n+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b)]\, d\theta \qquad \ldots (B.11)$$

$$\hat{\theta}_{ss2} = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c+0.5-0.5)}}{\Gamma(a+n+0.5c)} \theta^{-(2(a+n+0.5c-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b)]\, d\theta \qquad \ldots (B.12)$$

By multiplying the integral in equation (B.12) by the quantity which equals $B3 = (\frac{\Gamma(a+n+0.5c-0.5)}{\Gamma(a+n+0.5c-0.5)})$, where $\Gamma(.)$is a gamma function. Then we have

Appendix-B

The following is the derivation of these estimators under the squared error loss function.

The squared error loss function

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_1(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2$, the risk function is:

$$R(\hat{\theta}-\theta)=E[L_1(\hat{\theta},\theta)] \quad \text{...(B.1)}$$

$$R(\hat{\theta}-\theta)=\int_{\theta} L_1(\hat{\theta},\theta)\, P(\theta\setminus x)\, d\theta$$

$$R(\hat{\theta}-\theta)=\int_{\theta}(\hat{\theta}-\theta)^2\, P(\theta\setminus t)\, d\theta \quad \Rightarrow R(\hat{\theta}-\theta)=\int_{\theta}(\hat{\theta}^2-2\theta\hat{\theta}+\theta^2)\, P(\theta\setminus t)\, d\theta$$

$$R(\hat{\theta}-\theta)=\hat{\theta}^2\int_0^{\infty}P(\theta\setminus t)\, d\theta-2\hat{\theta}\int_0^{\infty}\theta\, P(\theta\setminus t)\, d\theta+\int_0^{\infty}\theta^2\, P(\theta\setminus t)\, d\theta \Rightarrow$$

$$R(\hat{\theta}-\theta)=\hat{\theta}^2-2\hat{\theta}E(\theta\setminus t)+E(\theta^2\setminus t) \quad \text{... (B.2)}$$

Let $\frac{\partial}{\partial\hat{\theta}}R(\hat{\theta}-\theta)=0$, we get Bayes estimator of θ denoted by $\hat{\theta}_{Bayes}$ for the above prior as follows

$$\hat{\theta}_{SE}=E(\theta\setminus x)=\int_0^{\infty}\theta\, P(\theta\setminus x)\, d\theta \quad \text{... (B.3)}$$

$$\hat{\theta}_{SE}=\int_0^{\infty}\theta\, P_i(\theta\setminus x)\, d\theta \quad ;\ i=1,2,3 \quad \text{... (B.4)}$$

1. **Bayes estimation using SRIG - natural conjugate family of priors distribution as double prior:**

To obtain the Bayes' estimator under SRIG - natural conjugate family of priors distribution as double prior. Substituting the equation (A.10) in equation (B.4), we get:

$$\hat{\theta}_{SE1}=\int_0^{\infty}\theta\, P_i(\theta\setminus x)\, d\theta \quad ;\ i=1 \quad \text{... (B.4)}$$

$$\hat{\theta}_{SE1}=\int_0^{\infty}\theta\frac{2(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)}\theta^{-(2(a+n+0.5\alpha+0.5)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)]\, d\theta \quad \text{... (B.5)}$$

$$\hat{\theta}_{SE1}=\int_0^{\infty}\frac{2(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)}\theta^{-(2(a+n+0.5\alpha+0.5-0.5)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)]\, d\theta \quad \text{... (B.6)}$$

$$\hat{\theta}_{SS1} = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)} \theta^{-(2(a+n+0.5\alpha)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta \quad \ldots (B.7)$$

By multiplying the integral in equation (B.7) by the quantity which equals to $B1 = (\frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha)})$, where $\Gamma(.)$ is a gamma function. Then we have

$$\hat{\theta}_{SS1} = B1\int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)} \theta^{-(2(a+n+0.5\alpha)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta \quad \ldots (B.8)$$

Then, we have

$$\hat{\theta}_{SS1} = \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{0.5} B2(t,\theta) \quad \ldots (B.9)$$

Where $B2(t,\theta)$ equals to

$$B2(t,\theta) = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{(a+n+0.5\alpha)}}{\Gamma(a+n+0.5\alpha)} \theta^{-(2(a+n+0.5\alpha)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)]\, d\theta = 1$$

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of $\hat{\theta}_{SS1}$ as the following formula:

$$\hat{\theta}_{SS1} = \frac{\Gamma(a+n+0.5\alpha)}{\Gamma(a+n+0.5\alpha+0.5)}(0.5\sum_{i=1}^{n} t_i^2 + b + \beta)^{0.5} \qquad n,\alpha,\beta,b,a > 0 \quad \ldots (B.10)$$

2. **<u>Bayes estimation using SRIG - non-informative distribution as double prior:</u>**

To obtain the Bayes' estimator under SRIG - non-informative_distribution as double prior. Substituting the equation (A.18) in equation (B.4), we get:

$$\hat{\theta}_{SS2} = \int_0^\infty \theta P_i(\theta \setminus x)\, d\theta \quad , \quad i = 2 \quad \ldots (B.4)$$

$$\hat{\theta}_{SS2} = \int_0^\infty \theta \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c)}}{\Gamma(a+n+0.5c)} \theta^{-(2(a+n+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b)]\, d\theta \quad \ldots (B.11)$$

$$\hat{\theta}_{SS2} = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c-0.5-0.5)}}{\Gamma(a+n+0.5c)} \theta^{-(2(a+n+0.5c-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b)]\, d\theta \quad \ldots (B.12)$$

By multiplying the integral in equation (B.12) by the quantity which equals $B3 = (\frac{\Gamma(a+n+0.5c-0.5)}{\Gamma(a+n+0.5c-0.5)})$, where $\Gamma(.)$ is a gamma function. Then we have

$$\hat{\theta}_{SRS2} = B3\int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c-0.5-0.5)}}{\Gamma(a+n+0.5c)} \theta^{-(2(a+n+0.5c-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b)]\, d\theta \qquad \ldots (B.13)$$

Then, we have

$$\hat{\theta}_{SRS2} = \frac{\Gamma(a+n+0.5c-0.5)}{\Gamma(a+n+0.5c)} (0.5\sum_{i=1}^{n} t_i^2 + b)^{0.5}\; B4(t,\theta) \qquad \ldots (B.14)$$

Where $B4(t,\theta)$ equals to

$$B4(t,\theta) = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + b)^{(a+n+0.5c-0.5)}}{\Gamma(a+n+0.5c-0.5)} \theta^{-(2(a+n+0.5c-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + b)]\, d\theta = 1$$

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of $\hat{\theta}_{SRS2}$ as the following formula:

$$\hat{\theta}_{SRS2} = \frac{\Gamma(a+n+0.5c-0.5)}{\Gamma(a+n+0.5c)} (0.5\sum_{i=1}^{n} t_i^2 + b)^{0.5} \qquad n, c, b, a > 0 \qquad \ldots (B.15)$$

3. **Bayes estimation using the natural conjugate family of priors- non-informative distribution as double prior:**

To obtain the Bayes' estimator under the natural conjugate family of priors- non-informative distribution, as double prior. Substituting the equation (A.24) in equation (B.4), we get:

$$\hat{\theta}_{SRS3} = \int_0^\infty \theta P_i(\theta \setminus x)\, d\theta \quad ; \quad i = 3 \qquad \ldots (B.4)$$

$$\hat{\theta}_{SRS3} = \int_0^\infty \theta \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n+0.5\alpha+0.5c)} \theta^{-(2(n+0.5\alpha+0.5c)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\, d\theta \qquad \ldots (B.16)$$

$$\hat{\theta}_{SRS3} = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c+0.5-0.5)}}{\Gamma(n+0.5\alpha+0.5c)} \theta^{-(2(n+0.5\alpha+0.5c-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\, d\theta \qquad \ldots (B.17)$$

By multiplying the integral in equation (B.17) by the quantity which equals to $B5 = (\frac{\Gamma(n+0.5\alpha+0.5c-0.5)}{\Gamma(n+0.5\alpha+0.5c-0.5)})$, where $\Gamma(.)$ is a gamma function. Then we have

$$\hat{\theta}_{SRS3} = B5\int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c+0.5-0.5)}}{\Gamma(n+0.5\alpha+0.5c)} \theta^{-(2(n+0.5\alpha+0.5c-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\, d\theta \qquad \ldots (B.18)$$

Then, we have

$$\hat{\theta}_{SRS3} = \frac{\Gamma(n+0.5\alpha+0.5c-0.5)}{\Gamma(n+0.5\alpha+0.5c)} (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{0.5}\; B_6(t,\theta) \qquad \ldots (B.19)$$

Where $B_6(t,\theta)$ equals to

$$B_6(t,\theta) = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c-0.5)}}{\Gamma(n+0.5\alpha+0.5c-0.5)} \theta^{-(2(n+0.5\alpha+0.5c-0.5)+1)} \exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\, d\theta = 1$$

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of $\hat{\theta}_{SRS3}$ as the following formula:

$$\hat{\theta}_{SRS3} = \frac{\Gamma(n+0.5\alpha+0.5c-0.5)}{\Gamma(n+0.5\alpha+0.5c)} (0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{0.5} \qquad ,\alpha, \beta, c, n \geq 0 \qquad \ldots (B.20)$$

Appendix-C: The following is the derivation of these estimators under the weighted squared error loss function.

1. **The weighted squared error loss function**

To obtain the Bayes' estimator, we minimize the posterior expected loss given by:

$L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta}$, the risk function is:

$$R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = E[L_2(\hat{\theta}, \theta)] \quad \ldots(C.1)$$

$$R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = \int_\theta L_2(\hat{\theta}, \theta)\, P(\theta \setminus t)\, d\theta \Rightarrow R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = \int_\theta \frac{(\hat{\theta} - \theta)^2}{\theta} P(\theta \setminus t)\, d\theta$$

$$R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = \int_\theta \frac{(\hat{\theta}^2 - 2\theta\hat{\theta} + \theta^2)}{\theta} P(\theta \setminus t)\, d\theta$$

$$R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = \int_\theta \left(\frac{\hat{\theta}^2}{\theta} - 2\hat{\theta} + \theta\right) P(\theta \setminus t)\, d\theta$$

$$R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = \hat{\theta}^2 \int_\theta \frac{1}{\theta} P(\theta \setminus t)\, d\theta - 2\hat{\theta} \int_\theta P(\theta \setminus t)\, d\theta + \int_\theta \theta P(\theta \setminus t)\, d\theta$$

$$R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = \hat{\theta}^2 E\left(\frac{1}{\theta} \setminus t\right) - 2\hat{\theta} + E(\theta \setminus t) \quad \ldots(C.2)$$

Let $\frac{\partial}{\partial \hat{\theta}} R_2\left(\frac{(\hat{\theta} - \theta)^2}{\theta}\right) = 0$, we get Bayes estimator of θ denoted by $\hat{\theta}_{wsz}$ for the above prior as follows

$$\hat{\theta}_{wszi} = \frac{1}{E\left(\frac{1}{\theta} \setminus t\right)} = \frac{1}{\int_0^\infty \frac{1}{\theta} P_i(\theta \setminus t)\, d\theta} \qquad i = 1,2,3 \quad \ldots(C.3)$$

1. **Bayes estimation using SRIG - natural conjugate family of priors distribution as double prior:**

To obtain the Bayes' estimator under SRIG - natural conjugate family of priors distribution as double prior. Substituting the equation (A.10) in the integral in equation (C.3), we get:

$$E\left(\frac{1}{\theta} \setminus t\right) = \int_0^\infty \frac{1}{\theta} P_1(\theta \setminus t)\, d\theta$$

$$E(\frac{1}{\theta}\backslash t)=\int_0^{\infty}\frac{1}{\theta}\frac{2(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)}\theta^{-(2(a+n+0.5\alpha+0.5)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)]\,d\theta \quad \ldots (C.4)$$

$$E(\frac{1}{\theta}\backslash t)=\int_0^{\infty}\frac{2(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)}\theta^{-(2(a+n+0.5\alpha+0.5+0.5)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)]\,d\theta \quad \ldots (C.5)$$

$$E(\frac{1}{\theta}\backslash t)=\int_0^{\infty}\frac{2(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)}}{\Gamma(a+n+0.5\alpha+0.5)}\theta^{-(2(a+n+0.5\alpha+1)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)]\,d\theta \quad \ldots (C.6)$$

By multiplying the integral in equation (C.6) by the quantity which equals to $C1=(\frac{\Gamma(a+n+0.5\alpha+1)}{\Gamma(a+n+0.5\alpha+1)})$, where $\Gamma(.)$ is a gamma function. Then we have

$$E(\frac{1}{\theta}\backslash t)=C1\int_0^{\infty}\frac{2(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{(a+n+0.5\alpha+0.5)+0.5-0.5}}{\Gamma(a+n+0.5\alpha+0.5)}\theta^{-(2(a+n+0.5\alpha+1)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)]\,d\theta \quad \ldots (C.9)$$

Then, we have

$$E(\frac{1}{\theta}\backslash t)=\frac{\Gamma(a+n+0.5\alpha+1)}{\Gamma(a+n+0.5\alpha+0.5)}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{-0.5}\,C2(t,\theta) \quad \ldots (C.10)$$

Where $C2(t,\theta)$ equals to

$$C2(t,\theta)=\int_0^{\infty}\frac{2(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{(a+n+0.5\alpha+1)}}{\Gamma(a+n+0.5\alpha+1)}\theta^{-(2(a+n+0.5\alpha+1)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)]\,d\theta=1.$$

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\frac{1}{\theta}\backslash t)=\frac{\Gamma(a+n+0.5\alpha+1)}{\Gamma(a+n+0.5\alpha+0.5)}(0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{-0.5} \quad \ldots (C.11)$$

Substituting the equation (C.11) in equation (C.3), we get:

$$\hat{\theta}_{WSE1}=\frac{\Gamma(a+n+0.5\alpha+0.5)\ (0.5\sum_{i=1}^{n}t_i^2+b+\beta)^{0.5}}{\Gamma(a+n+0.5\alpha+1)} \qquad n,\alpha,\beta,b,a>0 \quad \ldots (C.12)$$

2. **Bayes estimation using SRIG - non-informative distribution as double prior:**

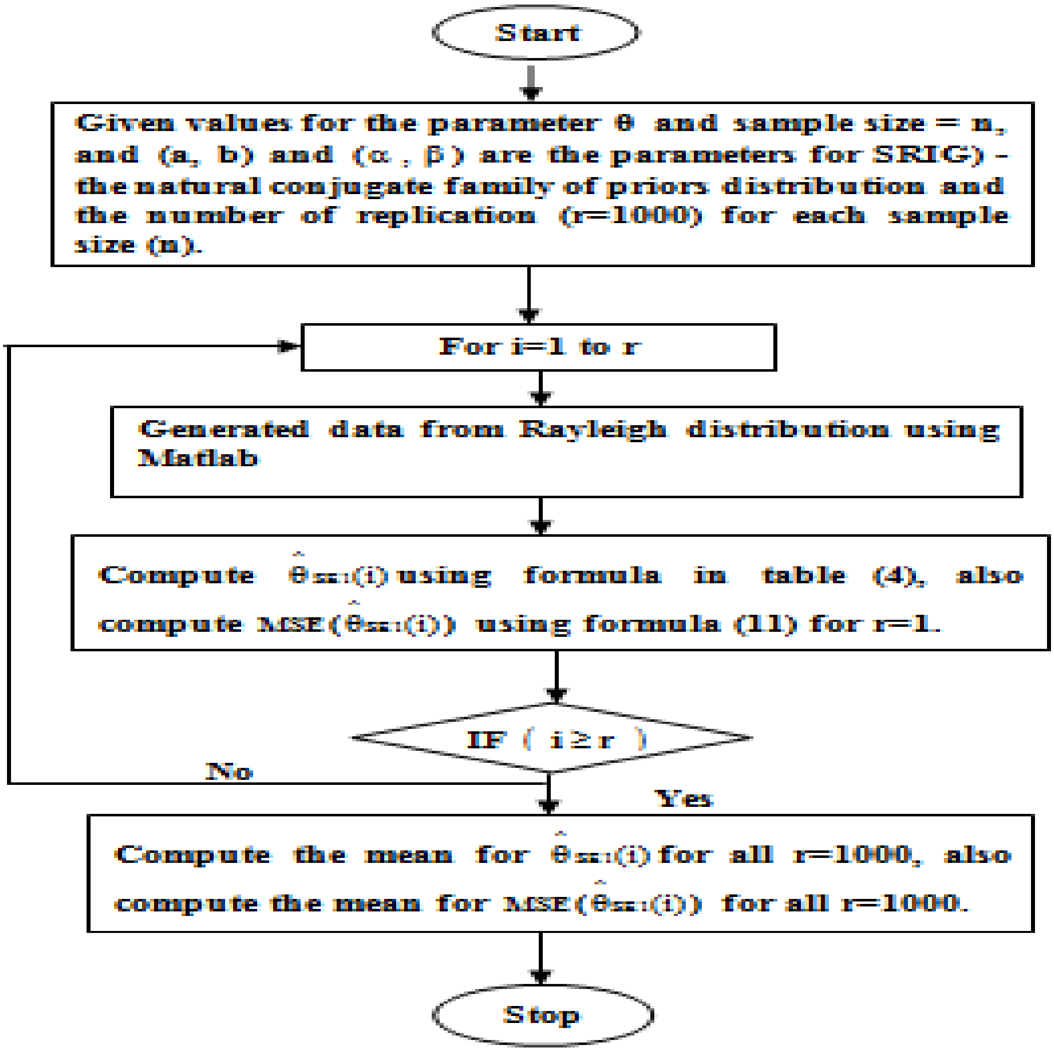
To obtain the Bayes' estimator under SRIG - non-informative distribution as double prior. Substituting the equation (A.18) in the integral in equation (C.3), we get:

$$E(\frac{1}{\theta}\backslash t)=\int_0^{\infty}\frac{1}{\theta}P_2(\theta\backslash t)\,d\theta$$

prior distribution for θ with MSE. The same thing we can do it to compute Bayes estimators $\hat{\theta}_{WSE}$.

<u>Appendix-D</u>: **The following is the programs algorithm.**

Algorithm (1): To compute Bayes estimators ($\hat{\theta}_{SE1}$) using SRIG - the natural conjugate family of priors Distribution as double prior distribution for θ with MSE.



Note (1): we can reformulate the Algorithm (1) to compute Bayes estimators $\hat{\theta}_{SEk}$, $k = 2,3$ using other distributions as double

$$E(\frac{1}{\theta}\backslash t) = \int_0^\infty \frac{1}{\theta}\frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)}}{\Gamma(n+0.5\alpha+0.5c)}\theta^{-(2(n+0.5\alpha+0.5c)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\,d\theta \quad \text{... (C.19)}$$

$$E(\frac{1}{\theta}\backslash t) = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c)+0.5-0.5}}{\Gamma(n+0.5\alpha+0.5c)}\theta^{-(2(n+0.5\alpha+0.5c+0.5)+1)}$$
$$\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\,d\theta \quad \text{... (C.20)}$$

By multiplying the integral in equation (C.20) by the quantity which equals to $C5 = (\frac{\Gamma(n+0.5\alpha+0.5c+0.5)}{\Gamma(n+0.5\alpha+0.5c+0.5)})$, where $\Gamma(.)$is a gamma function. Then we have

$$E(\frac{1}{\theta}\backslash t) = C5\int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c+0.5)-0.5}}{\Gamma(n+0.5\alpha+0.5c)}\theta^{-(2(n+0.5\alpha+0.5c+0.5)+1)}$$
$$\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\,d\theta \quad \text{... (C.21)}$$

Then, we have

$$E(\frac{1}{\theta}\backslash t) = \frac{\Gamma(n+0.5\alpha+0.5c+0.5)}{\Gamma(n+0.5\alpha+0.5c)}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{-0.5}\; C_6(t,\theta) \quad \text{... (C.22)}$$

Where $B_6(t,\theta)$ equals to

$$C_6(t,\theta) = \int_0^\infty \frac{2(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{(n+0.5\alpha+0.5c+0.5)}}{\Gamma(n+0.5\alpha+0.5c-0.5)}\theta^{-(2(n+0.5\alpha+0.5c+0.5)+1)}\exp[-\frac{1}{\theta^2}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)]\,d\theta = 1.$$

Be the integral of the pdf of SRIG distribution. Then we get the Bayes estimator of θ as the following formula:

$$E(\frac{1}{\theta}\backslash t) = \frac{\Gamma(n+0.5\alpha+0.5c+0.5)}{\Gamma(n+0.5\alpha+0.5c)}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{-0.5} \quad \text{... (C.23)}$$

Substituting the equation (C.23) in equation (C.3), we get:

$$\hat{\theta}_{WSE3} = \frac{\Gamma(n+0.5\alpha+0.5c)}{\Gamma(n+0.5\alpha+0.5c+0.5)}(0.5\sum_{i=1}^{n} t_i^2 + 0.5\beta)^{0.5} \quad ;\alpha,\beta,c,n \geq 0 \quad \text{... (C.24)}$$

Appendix-E: **The summarized and tabulated discussions and conclusions.**

Table 5-1: Best Estimation according to the smallest value for $MSE(\hat{\theta})$.

θ	The best estimation according to smallest value for $MSE(\hat{\theta})$ when the double prior distribution is	$MSE(\hat{\theta})$ Sample Size(n) 15	25	50	100
0.5	(SRIG - the natural conjugate family of priors) dist^n. with (a=5, b=0.5, α=8, β=0.5)	0.00276	0.00188	0.00103	0.00056
	(SRIG - the non-informative)) dist^n. with (a=5, b=0.5, c=5)	0.00345	0.00218	0.00111	0.00058
	(the natural conjugate family of priors- non-informative) dist^n. with(α=0.5, β=5, c=1)	0.02358	0.01021	0.00327	0.00109
1	(SRIG - the natural conjugate family of priors) dist^n. with (a=8, b=1, α=5, β=1)	0.01443	0.00893	0.00450	0.00238
	(SRIG - the non-informative)) dist^n. with (a=5, b=1, c=2)	0.01559	0.00942	0.00462	0.00239
	(the natural conjugate family of priors- non-informative) dist^n. with(α=0.5, β=5, c=1)	0.02103	0.01159	0.00518	0.00247
1.5	(SRIG - the natural conjugate family of priors) dist^n. with (a=8, b=1, α=5, β=1)	0.04622	0.0255	0.01162	0.00582
	(SRIG - the non-informative)) dist^n. with (a=5, b=1, c=2)	0.03949	0.02271	0.01079	0.00553
	(the natural conjugate family of priors- non-informative) dist^n. with(α=0.5, β=5, c=1)	0.03957	0.02304	0.01087	0.00544

Table5-2: Best Estimation according to the smallest value for $MWSE(\hat{\theta})$.

θ	The best estimation according to smallest value for $MWSE(\hat{\theta})$ when the double prior distribution is	$MWSE(\hat{\theta})$ Sample Size(n) 15	25	50	100
0.5	(SRIG - the natural conjugate family of priors) dist^n. with (a=5, b=0.5, α=8, β=0.5)	0.00558	0.00378	0.00206	0.00113
	(SRIG - the non-informative)) dist^n. with (a=5, b=1, c=5)	0.00643	0.00420	0.00218	0.00114
	(the natural conjugate family of priors- non-informative) dist^n. with(α=0.5, β=5, c=1)	0.04115	0.01828	0.00602	0.00206
1	(SRIG - the natural conjugate family of priors) dist^n. with (a=8, b=1, α=5, β=1)	0.01526	0.00925	0.00459	0.00241
	(SRIG - the non-informative)) dist^n. with (a=5, b=1, c=2)	0.01573	0.00944	0.00463	0.00240
	(the natural conjugate family of priors- non-informative) dist^n. with (α=0.5, β=5, c=1)	0.01818	0.01052	0.00491	0.00242
1.5	(SRIG - the natural conjugate family of priors) dist^n. with (a=8, b=1, α=5, β=1)	0.03352	0.01815	0.00808	0.00399
	(SRIG - the non-informative)) dist^n. with (a=5, b=1, c=2)	0.02782	0.01570	0.00734	0.00374
	(the natural conjugate family of priors- non-informative) dist^n. with(α=0.5, β=5, c=1)	0.02442	0.01460	0.00705	0.00359

A Comparison of Bayes Estimators for the parameter of Rayleigh Distribution with Simulation

مقارنة لمقدرات بيز لمعلمة توزيع رايلي باستعمال المحاكاة

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الكلية التقنية الإدارية – بغداد

المستخلص

مقارنـة للتوزيعـات الاوليـة المضـاعفة مفترضـة لمعلمـة توزيـع رايلـي .تضـمنة ثـلاث مجـاميع للتوزيعـات المضـاعفة للمعلمـة المجهولـة لتوزيـع رايلـي. فقـد تـم افتـراض ثلاثـة توزيعـات اوليـة مضـاعفة : توزيـع الجذرالتربيعي لمعكوس كاما (SRIG)- الاولي العائلة المرافقة الطبيعية ، توزيع الجذرالتربيعي لمعكوس كاما (SRIG)- التوزيـع غيـر المعلومـاتي ، توزيـع العائلـة المرافقـة الطبيعيـة - التوزيـع غيـر المعلومـاتي، كـدوال معلوماتيـة مضـاعفه. البيانـات ولـدت مـن ثلاثـة حـالات لتوزيـع رايلـي لحجـوم مختلفـة مـن العينـات (الصـغيرة والمتوسطة والكبيرة) . وتم اشتقاق مقدرات بيز للمعلمة باستعمال دالـة الخسـارة التربيعيـة والدالـة التربيعيـة الموزونه مع ثلاثة دوال معلوماتية مضاعفة. استعملت المحاكاة للحصول على نتائج هذا البحث. وتحديد افضل مقدر وفقا لاقل قيمة لمعيار متوسط مربعات الخطأ ومتوسط مربعات الخطأ الموزونـة. وجدنا بـان افضل تقدير للمعلمة (θ) لكل حجوم العينات ، عندما يكون التوزيع الاولـي المضـاعف توزيـع الجذرالتربيعي لمعكوس كاما (SRIG)- الاولـــي العائلـــة المرافقـــة الطبيعيـــة عنـــد القـــيم ($a = 5, b = 0.5, \alpha = 8, \beta = 0.5$) و ($a = 8, b = 1, \alpha = 5, \beta = 1$) للقيمتـين الحقيقيـة للمعلمـة ($\theta = 0.5 , 1$) وعلـى التـوالي . كـذلك حصـلنا على افضل تقدير لـ θ عندما يكون التوزيع الاولي المضاعف توزيع العائلة المرافقة الطبيعيـة - التوزيـع غير المعلوماتي عند القيم ($\alpha = 0.5, \beta = 5 , c = 1$) للقيمة الحقيقية $\theta = 1.5$.

المصطلحات الرئيسة للبحث/ توزيـع رايلـي، طريقـة بيـز، التوزيـع الاولـي معلوماتيـة والغيـر المعلوماتيـة المضاعفه ، التوزيع الاحق، دالة الخسارة التربيعية ، دالـة الخسـارة التربيعيـة الموزونـة .