On Shrunken Estimation of Generalized Exponential Distribution

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حول التقدير المتقلص للتوزيع الاسي العام

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المستخلص:

يتعامل موضوع هذا البحث مع تقدير معلمة الشكل (α) للتوزيع الاسي العام ذو المعلمتين عندما تكون معلمة القياس (λ) معلومة باستخدام مقدر الاختبار الاولي المتقلص ذو المرحلة الواحدة (SSSE) عند توافر معلومات مسبقة (α_0) حول معلمة الشكل (α) بشكل قيمه ابتدائية عن طريق الخبرات السابقة بالاضافة الى منطقة اختبار (R) لهذه المعلومات المسبقة.

اشتقت معادلات التحيز، متوسط مربعات الخطأ [MSE] والكفاءة النسبية [R.Eff(·)] للمقدر المقترح. نوقشت خصائص المقدر المقترح عن طريق دراسة النتائج العددية الخاصة بالمعادلات المذكورة آنفاً ووضعت هذه النتائج بشكل جداول مرفقة.

اجريت بعض المقارنات بين المقدر المقترح والمقدر الكلاسيكي بالاضافة الى المقارنة مع بعض الدراسات السابقة لبيان أهمية وكفاءة المقدر المقترح.

Abstract:

This paper deal with the estimation of the shape parameter (α) of Generalized Exponential (GE) distribution when the scale parameter (λ) is known via preliminary test single stage shrinkage estimator (SSSE) when a prior knowledge (α_0) a vailable about the shape parameter as initial value due past experiences as well as suitable region (R) for testing this prior knowledge.

The Expression for the Bias, Mean squared error [MSE] and Relative Efficiency [R.Eff(·)] for the proposed estimator are derived. Numerical results about behavior of considered estimator are discussed via study the mentioned expressions. These numerical results displayed in annexed tables. Comparisons between the proposed estimator and the classical estimator as well as with some earlier studies were made to shown the effect and usefulness of the considered estimator.

1. Introduction

The two parameter generalized exponential (GE) distribution has been proposed and studied by the authors. It has been studied extensively by Gupta and kundu [1], [2], [3], [4], [5] and [6].

Note that the Generalized Exponential (GE) distribution is a sub-module of the exponentiated weibull distribution introduced by [7].

It is observed in [1] that the two parameter Generalized Exponential (GE) distribution can be used quite effectively in analyzing many lifetime skewed data, and the properties of GE distribution are quite close to corresponding properties of two parameter Gamma distribution.

The two parameter Generalized Exponential (GE) distribution has the following distribution function:

$$F(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}\ ;\ x > 0,\ \alpha,\ \lambda > 0 \qquad \ldots(1)$$

Therefore, GE distribution has the following density function

$$f(x;\alpha,\lambda) = \begin{cases} \alpha\lambda(1-e^{-\lambda x})^{\alpha-1} e^{-\lambda x} & \text{for } x > 0, \alpha, \lambda > 0 \\ 0 & \text{o.w.} \end{cases} \qquad \ldots(2)$$

Here, α is the shape parameter and λ is the scale parameter.

We denoted by GE(α,λ) to Generalized Exponential distribution with shape parameter α and scale parameter λ.

In this paper we introduce the problem for estimating the unknown shape parameter (α) of GE distribution with known scale parameter (λ) when some prior knowledge (α_0) regarding true value (α) is a vailable using preliminary test single stage shrinkage procedure.

Noted that, the prior knowledge regarding due reasons introduced by [8] as well as the classical estimator of α ($\hat{\alpha}$;MLE) and using shrinkage weight factor [$\psi_1(\hat{\alpha})$], $0 \le \psi_1(\hat{\alpha}) \le 1$ results the what is known as "Shrinkage estimator", which though perhaps biased has smaller mean squared error [MSE] than that of $\hat{\alpha}$.

Thus, "Thompson-Type" Shrinkage estimator will be

$$\psi_1(\hat{\alpha})\hat{\alpha} + (1-\psi_1(\hat{\alpha}))\alpha_0 \qquad \text{...(3)}$$

Now, the preliminary test single stage shrinkage estimator (SSSE) introduced in this paper is a estimator of level of significance (Δ) for test the hypotheses $H_0:\alpha = \alpha_0$ vs $H_A: \alpha \neq \alpha_0$.

If H_0 accepted we use the shrinkage estimator defined in (3).

However, if H_0 rejected, we shall take another shrinkage estimator via different shrinkage weight factor $\psi_2(\cdot)$; $0 \le \psi_2(\cdot) \le 1$ and then using the following shrinkage estimator:

$$\psi_2(\hat{\alpha})\hat{\alpha} + (1-\psi_2(\hat{\alpha}))\alpha_0 \qquad \text{...(4)}$$

Thus, the general form of preliminary test single stage Shrinkage estimator (SSSE) will be:

$$\tilde{\alpha} = \begin{cases} \psi_1(\hat{\alpha})\hat{\alpha} + (1-\psi_1(\hat{\alpha}))\alpha_0 & ,\text{if } \hat{\alpha} \in R \\ \psi_2(\hat{\alpha})\hat{\alpha} + (1-\psi_2(\hat{\alpha}))\alpha_0 & ,\text{if } \hat{\alpha} \notin R \end{cases} \qquad \text{...(5)}$$

where $\psi_i(\hat{\alpha}), 0 \le \psi_i(\hat{\alpha}) \le 1$, i = 1, 2 is a shrinkage weight factor specifying the belief of $\hat{\alpha}$ and $(1-\psi_i(\hat{\alpha}))$ specifying the belief of α_0 and $\psi_i(\hat{\alpha})$ may be a function of $\hat{\alpha}$ or may be a constant (ad hoc basis), while (R) is a pretest region for acceptance the prior knowledge with level of significance Δ.

Several authors have been studied preliminary test single stage shrinkage estimator (SSSE) defined in (5), see for example; [8], [9], [10], [11] and [12].

The aim of this paper is to estimate the shape parameter (α) of two parameters Generalized Exponential (GE) distribution with known scale parameter ($\lambda = 1$) using proposed preliminary test (SSSE) defined in (5) via study the expressions of Bias, Mean squared error and Relative Efficiency of this estimator and display the numerical results for mentioned expressions in annexed tables. Also, study the performance of the consider estimator and make comparisons with the classical estimator as well as with some studies introduced by some authors.

2. Maximum Likelihood Estimator of α[2]

In this section, we consider the maximum likelihood estimator (MLE) of GE(α,λ). Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from G (α,λ), then the log-likelihood function L(α,λ) can be written as:

$$L(\alpha,\lambda) = n\ln(\alpha) + n\ln(\lambda) + (\alpha-1)\sum_{i=1}^{n}\ln(1-e^{-\lambda x_i}) - \lambda\sum_{i=1}^{n}x_i \qquad \text{...(6)}$$

In this paper, we assume that λ is known ($\lambda = 1$).

The normal equation become

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-e^{-x_i}) = 0 \qquad \text{...(7)}$$

Thus , we obtain the MLE of α, say $\hat{\alpha}_{MLE}$ as below

$$\hat{\alpha}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln(1-e^{-x_i})} \qquad \ldots(8)$$

The distribution of $\hat{\alpha}_{MLE}$ is the same as the distribution of (nα/y), where y follows Gamma (n,1); see [2].
Therefore, for n > 2

$$E(\hat{\alpha}_{MLE}) = \frac{n}{n-1}\alpha, \ \text{var}(\hat{\alpha}_{MLE}) = \frac{n^2\alpha^2}{(n-1)^2(n-2)} \ \textbf{and} \ \text{MSE}(\hat{\alpha}_{MLE}) = \frac{(n+2)\alpha^2}{(n-1)(n-2)}$$

Now, using (8), an unbiased estimator of α can be easily obtained as:

$$\hat{\alpha} = \frac{n-1}{n}\hat{\alpha}_{MLE} = -\frac{(n-1)}{\sum_{i=1}^{n} \ln(1-e^{-x_i})} \qquad \ldots(9)$$

Therefore, $E(\hat{\alpha}) = \alpha$ **and** $\text{MSE}(\hat{\alpha}) = \text{var}(\hat{\alpha}) = \frac{\alpha^2}{n-2}$ **…(10)**

3. Preliminary Test Single Stage Shrinkage Estimator (SSSE) $\tilde{\alpha}$

In this section, we consider the preliminary test (SSSE) which is defined in (5) when $\psi_1(\hat{\alpha}) = 0$ **and** $\psi_2(\hat{\alpha}) = w(p)$ **for estimate the shape parameter α of GE distribution when λ = 1.**
Thus, preliminary test single stage shrinkage estimator (SSSE) $\tilde{\alpha}$ **will be::**

$$\tilde{\alpha} = \begin{cases} \alpha_0 & ,\text{if } \hat{\alpha} \in R \\ w(p)\hat{\alpha} + (1-w(p))\alpha_0 & ,\text{if } \hat{\alpha} \notin R \end{cases} \qquad \ldots(11)$$

where w(p) = $k_1(p)$ / $k_2(p)$

$$k_i(p) = \left(\frac{2}{n-2}\right)^{ip} \cdot \Gamma\left(\frac{n+2ip-2}{2}\right) \Big/ \Gamma\left(\frac{n-2}{2}\right)$$ **, for i = 1, 2, p ∈ N and n > 2.**

And R is a pretest region for testing the hypothesis H_0: $\alpha = \alpha_0$ vs H_A: $\alpha \neq \alpha_0$ with level of significance (Δ) using test statistic $T(\hat{\alpha}/\alpha_0) = \frac{2(n-1)\alpha_0}{\alpha}$**.**

i.e.; $R = \left[\frac{2(n-1)\alpha_0}{b}, \frac{2(n-1)\alpha_0}{a}\right]$ **…(12)**

Where, $a = (X^2_{1-\Delta/2,2n})$ **and** $b = (X^2_{\Delta/2,2n})$, **…(13)**

are respectively the lower and upper 100(Δ/2) percentile point of chi-square distribution with degree of freedom 2n.

The expression for Bias of $\tilde{\alpha}$ **is**

$$\text{Bias}(\tilde{\alpha}/\alpha, R) = E(\tilde{\alpha} - \alpha)$$

$$= \int_R (\alpha_0 - \alpha) f(\hat{\alpha}) d\hat{\alpha} + \int_{\bar{R}} [w(p)\hat{\alpha} + (1 - w(p))\alpha_0 - \alpha] f(\hat{\alpha}) d\hat{\alpha}$$

Where, $\bar{R}$ **is the complement region of R in real space and f(**$\hat{\alpha}$**) is a p.d.f. of** $\hat{\alpha}$ **with the following form**

$$f(\hat{\alpha}) = \begin{cases} \dfrac{\left[\dfrac{(n-1)}{\hat{\alpha}}\alpha\right]^{n+1} e^{\frac{-(n-1)\alpha}{\hat{\alpha}}}}{\Gamma(n)(n-1)\alpha} & \text{for } \hat{\alpha} > 0, \alpha > 0 \\ 0 & \text{o.w.} \end{cases} \quad \textbf{...(14)}$$

We conclude,

$$\text{Bias}(\tilde{\alpha}/\alpha, R) = \alpha\left\{(\zeta - 1)J_0(a^*, b^*) + (1 - w(p))(\zeta - 1) - (n-1)w(p)J_1(a^*, b^*) - (1 - w(p))\zeta J_0(a^*, b^*) + J_0(a^*, b^*)\right\}$$

…(15)

where $J_\ell(a^*, b^*) = \int_{a^*}^{b^*} y^{-\ell} \dfrac{y^{n-1} e^{-y}}{\Gamma(n)} dy; \ell = 0, 1, 2$ **…(16)**

Also, $\zeta = \dfrac{\alpha_0}{\alpha}, y = \dfrac{(n-1)\alpha}{\hat{\alpha}}, a^* = \zeta^{-1} \cdot a$ **and** $b^* = \zeta^{-1} \cdot b$ **…(17)**

The Bias ratio [B(·)] of $\tilde{\alpha}$ **is defined as below:-**

$$B(\tilde{\alpha}) = \frac{\text{Bias}(\tilde{\alpha}/\alpha, R)}{\alpha} \quad \textbf{...(18)}$$

The expression of Mean squared error (MSE) of $\tilde{\alpha}$ **given as**

$$\text{MSE}(\tilde{\alpha}/\alpha, R) = E(\tilde{\alpha} - \alpha)^2$$

$$= \alpha^2 \Big\{ (\zeta - 1)^2 J_0(a^*, b^*) + \frac{[w(p)]^2}{n-2} + (\zeta - 1)^2 (w(p) - 1)^2 - [w(p)]^2 [(n-1)^2 J_2(a^*, b^*) -$$

$$2(n-1)\zeta J_1(a^*, b^*) + \zeta^2 J_0(a^*, b^*)] - 2w(p)(\zeta - 1)[(n-1)J_1(a^*, b^*) - \zeta J_0(a^*, b^*)] - (\zeta - 1)^2 J_0(a^*, b^*) \Big\}$$

…(19)

The Efficiency of $\tilde{\alpha}$ **relative to the** $\hat{\alpha}$ **denoted by R.Eff(**$\tilde{\alpha}$**/α,R) defined as**

$$\text{R.Eff}(\tilde{\alpha}/\alpha, R) = \frac{\text{MSE}(\hat{\alpha})}{\text{MSE}(\tilde{\alpha}/\alpha, R)} \quad \textbf{...(20)}$$

See for example; [8], [9], [10] and [11].

4. Conclusions and Numerical Results

The computations of Relative Efficiency [R.Eff(·)] and Bias Ratio [B(·)] expression were used for the considered estimators $\tilde{\alpha}$ (using MATH.Cad 2001 program) . These computations were performed for the constants Δ = 0.05,0.01,0.1, n = 4,6,8,10,12,16,20,30, p = 2,3,4,5 and ζ = 0.25(0.25), 2. Some of these computations are displayed in tables (1) and (2) for some samples of these constants. The observation mentioned in the tables leads to the following results:

- **i. The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ are adversely proportional with small value of Δ, i.e. Δ = 0.01 yield highest efficiency.**
- **ii. The Relative Efficiency [R.Eff (·)] are increasing function with increasing value of p.**
- **iii. The Relative Efficiency [R.Eff (·)] of $\tilde{\alpha}$ has maximum value when $\alpha=\alpha_0(\zeta=1)$, for each p, n,Δ, and decreasing otherwise ($\zeta\neq1$). This feature shown the important usefulness of prior knowledge which given higher effects of proposed estimator as well as the important role of shrinkage technique and its philosophy.**
- **iv. Bias ratio [B(·)] of $\tilde{\alpha}$ increases when ζ increases.**
- **v. Bias ratio [B(·)] of $\tilde{\alpha}$ are reasonably small when $\alpha = \alpha_0$ for each w(p), n, Δ, p and increases otherwise. This property shown that the proposed estimator $\tilde{\alpha}$ is very closely to unbiasedness especially when $\alpha = \alpha_0$.**
- **vi. The Relative efficiency [R.Eff(·)] of $\tilde{\alpha}$ decreases function with increases value of w(p) and n, for each p, Δ, ζ. This property employ the role of the prior information for proposed Shrinkage estimator via takes high weight for prior information which leads to maximum efficiency.**
- **vii. The Effective Interval [the value of ζ that makes R.Eff. (·) greater than one] using proposed estimator $\tilde{\alpha}$ is at most [0.5, 1.25]. Here the pretest criterion is very important for guarantee that prior information is very closely to the actual value and prevents it faraway from it, which get optimal effect of the considered estimator to obtain high efficiency.**
- **viii. The considered estimator $\tilde{\alpha}$ is better than the classical estimator especially when $\alpha \approx \alpha_0$, which is given the effective of $\tilde{\alpha}$ when given an important weight of prior knowledge. And the augmentation of efficiency may be reach to tens times.**
- **ix. The proposed estimator $\tilde{\alpha}$ has smaller MSE than some existing estimators introduced by authors, see for examples [2].**

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Table (1) Shown Bias Ratia B(·) and R.E.ff of $\tilde{\alpha}$ w.r.t. Δ, n and ζ when p = 3

			ζ					
Δ	**n**	**R.Eff. Bias**	**0.25**	**0.75**	**1**	**1.25**	**1.75**	**2**
0.01	**4**	**R.Eff(·) B(·)**	**1.0071055 (– .70286)**	**8.5716 (– .237567)**	**347.5272 (.000702)**	**2.6329 (.293074)**	**.883657 (.751784)**	**0.4989075 (1.000803)**
	8	**R.Eff(·) B(·)**	**0.349546 (– .689732)**	**3.081120 (– .2304131)**	**238.989 (.008859)**	**2.569178 (.254026)**	**0.295571 (.750838)**	**0.166691 (.9998513)**
	16	**R.Eff(·) B(·)**	**.228255 (– .5550749)**	**1.913522 (– .1817786)**	**21.6614 (0.019417)**	**1.05324 (.258693)**	**0.126732 (.7506336)**	**.007161212 (0.998321)**
	20	**R.Eff(·) B(·)**	**.2171433 (– .4996574)**	**1.694985 (– .164432)**	**14.29253 (.0028292)**	**.832049 (.256508)**	**.009874 (.74991)**	**.0055757 (.997582)**
0.05	**4**	**R.Eff(·) B(·)**	**1.0153918 (– .7001389)**	**9.1488148 (– 0.229575)**	**286.32138 (.0129981)**	**7.33851 (0.2585977)**	**0.879730 (0.753276)**	**0.499158 (1.000337)**
	8	**R.Eff(·) B(·)**	**0.349549 (– .6897299)**	**3.179488 (– 0.22696)**	**221.442999 (0.0119664)**	**2.5272137 (0.256090)**	**.295485 (.7508839)**	**.16732071 (.99781011)**
	16	**R.Eff(·) B(·)**	**.228255 (– .555074)**	**1.864801 (–.183649)**	**21.661428 (.01941726)**	**1.112045 (.250876)**	**.1279409 (.7466908)**	**.07266155 (.98970729)**
	20	**R.Eff(·) B(·)**	**.217149 (– .49965)**	**1.650754 (– .1660671)**	**12.19599 (.015806)**	**.9352259 (0.239371)**	**.1003849 (.743093)**	**.0568135 (0.98604029)**

Table (2) Shown Bias Ratia B(·) and R.E.ff of $\tilde{\alpha}$ w.r.t. Δ, n and ζ when p = 4

			ζ					
Δ	**n**	**R.Eff. Bias**	**0.25**	**0.75**	**1**	**1.25**	**1.75**	**2**
0.01	**4**	**R.Eff(·) B(·)**	**.88989 (– 0.749575)**	**8.007153 (– .249887)**	**428188 (.0000633)**	**7.997506 (.2500388)**	**0.88885 (.75001)**	**0.49999 (1.000007)**
	8	**R.Eff(·) B(·)**	**.3060477 (– .73792)**	**2.750568 (–.246075)**	**5953.9753 (.0017749)**	**2.648961 (.2508066)**	**.296161 (.7501680)**	**.166675 (.99997)**
	16	**R.Eff(·) B(·)**	**.1564909 (– .675074)**	**1.408589 (– .223777)**	**157.3529 (.01022725)**	**1.1106166 (.2533417)**	**.1268961 (.7502435)**	**.071512 (.99935)**
	20	**R.Eff(·) B(·)**	**.136343 (– .6373499)**	**1.2105582 (– .2114960)**	**70.585589 (.0127311)**	**.8657561 (.2529288)**	**.98760 (.74999)**	**.055662 (.99891)**
0.05	**4**	**R.Eff(·) B(·)**	**.89011 (– .749483)**	**8.013533 (– .249788)**	**2667497 (.000134)**	**7.99428 (.250089)**	**.888808 (.750039)**	**.499996 (1.000003)**
	8	**R.Eff(·) B(·)**	**.306048 (– 0.737924)**	**2.766229 (– .24538)**	**5516.8358 (.002397)**	**2.64020 (.251220)**	**.296151 (.7501770)**	**.166809 (0.999561)**
	16	**R.Eff(·) B(·)**	**.15649 (– 0.675074)**	**1.39851 (– .22449)**	**146.6117 (.007463)**	**1.13631 (.250337)**	**.127390 (.748728)**	**.07195 (.99604)**
	20	**R.Eff(·) B(·)**	**.13634 (– .63735)**	**1.200603 (– .212232)**	**60.23516 (.0071124)**	**.245217 (.91739)**	**.099544 (.7468922)**	**.056195 (.993718)**